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Quantum Field Theory in Curved Spacetime
Problem Sheet 7

Problem 7.1 (GNS construction) [5 points]

1. Let (Ω, μ) be a probability space, i.e., a measure space Ω and a probability (i.e., positive and normalized) measure μ. Consider the algebra $\mathfrak{A} = C(\Omega, \mu, \mathbb{C})$, i.e., the complex measurable functions on (Ω, μ) with adjoint given by complex conjugation. Show that

$$\omega(f) = \int_\Omega f(x) d\mu(x)$$

defines a state and perform the GNS construction.

2. Consider the algebra $\mathfrak{A} = \mathrm{Mat}(n \times n, \mathbb{C})$ of complex $n \times n$ matrices. Consider the functionals

$$\begin{aligned} \omega_1(A) &= A_{11}, \\ \omega_2(A) &= A_{11} + A_{12} + A_{21}, \\ \omega_3(A) &= \frac{1}{n} \operatorname{tr} A. \end{aligned}$$

Which of these are states? Which of those are pure? Perform the GNS construction on the states and decompose into irreducibles, if possible.

Problem 7.2 (Quasifree states) [7 points]

The observable algebra for a quantum mechanical single particle in 1 dimension can be thought of as generated by $\hat{\phi}(q,p)$, with $q, p \in \mathbb{C}$, subject to the relations

$$\begin{aligned} \hat{\phi}(\lambda q, \lambda p) &= \lambda \hat{\phi}(q,p), \\ \hat{\phi}(q_1 + q_2, p_1 + p_2) &= \hat{\phi}(q_1, p_1) + \hat{\phi}(q_2, p_2), \\ \hat{\phi}(q,p)^* &= \hat{\phi}(\bar{q}, \bar{p}), \\ [\hat{\phi}(q_1,p_1), \hat{\phi}(q_2,p_2)] &= iE((q_1,p_1),(q_2,p_2)) = i(q_1 p_2 - p_1 q_2). \end{aligned}$$

We may think of the position and momentum operator as given by $\hat{q} = \hat{\phi}(1,0)$, $\hat{p} = \hat{\phi}(0,1)$. Consider quasi-free states of the form

$$\omega(e^{i\hat{\phi}(q,p)}) = e^{-\mu((q,p),(q,p))/2}$$

with

$$\mu((q_1,p_1),(q_2,p_2)) = \lambda q_1 q_2 + \eta p_1 p_2.$$

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1. Show that the positivity condition

$$\mu((q_1,p_1),(q_1,p_1))\mu((q_2,p_2),(q_2,p_2)) \geq \frac{1}{4}E((q_1,p_1),(q_2,p_2))^2, \qquad q_i,p_i \in \mathbb{R},$$

   is fulfilled iff $\lambda\eta \geq \frac{1}{4}$. Show that the latter condition is equivalent to the fulfillment of Heisenberg's uncertainty relation. [Note that $\hbar = 1$.]

2. Show that the state is pure, i.e., fulfills

$$\mu((q_1,p_1),(q_1,p_1)) = \frac{1}{4}\sup_{(q_2,p_2)\neq 0} \frac{E((q_1,p_1),(q_2,p_2))^2}{\mu((q_2,p_2),(q_2,p_2))}, \qquad q_i,p_i \in \mathbb{R},$$

   iff $\lambda\eta = \frac{1}{4}$.

3. For the pure states, perform explicitly the construction of the one-particle Hilbert space $\mathcal{H}_1$. Show that it is one-dimensional, i.e., $\mathcal{H}_1 \simeq \mathbb{C}$ and determine $\pi(\hat{q})$ and $\pi(\hat{p})$ in terms of the creation and annihilation operators a^* and a.