UNIVERSITÄT LEIPZIG

Institut für Theoretische Physik

Prof. Dr. R. Verch

Dr. J. Zahn

Summer Term 2019

Quantum Field Theory in Curved Spacetime
Problem Sheet 6

Problem 6.1 [5 points]

Let (M, g_{ab}) be a globally hyperbolic spacetime, and let φ and ψ be real-valued, C^∞ solutions to the Klein-Gordon equation $(\nabla^a\nabla_a - m^2)\chi = 0$ having compactly supported Cauchy data.

Show that the symplectic form

$$\sigma(\varphi, \psi) = \int_S (\varphi n^a \nabla_a \psi - \psi n^a \nabla_a \varphi)\, d|\mathrm{vol}_S|$$

is independent of the Cauchy-surface S (i.e. it has the same value for any Cauchy-surface S.

Hint: You may use the following generalization of Gauss' theorem: Suppose that U is an open subset of M with a piecewise smooth boundary ∂U so that on the smooth pieces, the outward pointing unit normal field ν^a is defined (meaning $g_{ab}\nu^a\nu^b = \pm 1$) as well as the metric-induced measure $d|\mathrm{vol}_{\partial U}|$. Then for any smooth vector field V^a it holds that

$$\int_U \nabla_a V^a\, d|\mathrm{vol}_g| = \int_{\partial U} g_{ab} V^a \nu^b\, d|\mathrm{vol}_{\partial U}|$$

provided the integrals exist. The vector field to consider here is $V^a = \varphi\nabla^a\psi - \psi\nabla^a\varphi$.

Problem 6.2 [7 points]

Let Σ be a 3-dimensional smooth manifold with a Riemannian metric $\bar{g}^{(\Sigma)}_{ab}$ and the associated Laplace operator $\Delta = \bar{g}^{(\Sigma)ab}\bar{\nabla}_a\bar{\nabla}_b$ on functions $f \in C_0^\infty(M, \mathbb{C})$. Consider the ultrastatic spacetime (M, g_{ab}) given as

$$M = \mathbb{R} \times \Sigma\,, \qquad g_{ab} = -dt_a\, dt_b + \bar{g}^{(\Sigma)}_{ab}$$

Suppose that the hypersurfaces $\Sigma_t = \{t\} \times \Sigma$ ($t \in \mathbb{R}$) are Cauchy-surfaces.

(i) Show that the Klein-Gordon operator $\mathcal{K} = \nabla^a\nabla_a - m^2$ on $C^\infty(M, \mathbb{C})$ takes the form

$$\mathcal{K}\varphi(t, \boldsymbol{\sigma}) = -\frac{\partial^2}{\partial t^2}\varphi(t, \boldsymbol{\sigma}) + (\Delta - m^2)\varphi(t, \boldsymbol{\sigma})$$

(note that Δ acts only with respect to $\boldsymbol{\sigma} \in \Sigma$).

/...2

(ii) Define the operator $A = -(\Delta - m^2)$ on $C_0^\infty(\Sigma, \mathbb{C})$. This is a densely defined, symmetric, positive (and even essentially selfadjoint) linear operator in the Hilbert space $L^2(\Sigma, dvol_\Sigma)$ where $dvol_\Sigma$ is the measure induced by the Riemannian metric $\bar{g}_{ab}^{(\Sigma)}$ on Σ.

Show that for given functions $u, v \in C_0^\infty(\Sigma, \mathbb{C})$ the function

$$\varphi(t, \boldsymbol{\sigma}) = \cos(t\sqrt{A})u(\boldsymbol{\sigma}) + \frac{\sin(t\sqrt{A})}{\sqrt{A}}v(\boldsymbol{\sigma})$$

is a solution to the Klein-Gordon equation, $\mathcal{K}\varphi(t, \boldsymbol{\sigma}) = 0$, having the Cauchy data

$$\varphi(0, \boldsymbol{\sigma}) = u(\boldsymbol{\sigma}), \qquad \frac{\partial}{\partial t}\varphi(t, \boldsymbol{\sigma})\bigg|_{t=0} = v(\boldsymbol{\sigma})$$

Hint: Operators $f(A)$ are defined by the functional calculus. E.g. in the case that the manifold Σ is compact, the operator A possesses an orthonormal basis $\{\psi_k\}_{k\in\mathbb{N}}$ of eigenvectors (all contained in $C^\infty(\Sigma, \mathbb{C})$) with corresponding eigenvalues $\lambda_k > 0$, and in this case, $f(A)\chi = \sum_k f(\lambda_k)(\psi_k, \chi)\psi_k$ for any $\chi \in C^\infty(\Sigma, \mathbb{C})$.