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Quantum Field Theory in Curved Spacetime
Problem Sheet 3

Problem 3.1 [7 points]

(A)

Discuss if the following spacetimes are globally hyperbolic (showing the existence of a Cauchy surface is sufficient for proving global hyperbolicity). If they are globally hyperbolic, provide a smooth slicing into Cauchy surfaces.

(a) $I^+(0)$ in $1+d$-dimensional Minkowski spacetime

(b) $M = \mathsf{M}\backslash D$, where M is Minkowski spacetime and $D = \overline{\mathcal{O}(x,y)}$ is a (non-empty) *closed* double cone

(c) de Sitter spacetime, which is the 4-dimensional submanifold of 5-dimensional Minkowski spacetime given by

$$dS = \{(x^0, \boldsymbol{x}, w) \in \mathbb{R}^5 : (x^0)^2 - |\boldsymbol{x}|^2 - w^2 = -1\}$$

and equipped with the metric inherited by restriction of the ambient 5-dimensional Minkowski metric to the submanifold, including the inherited time-orientation.

(d) Anti-de Sitter spacetime, which is the 4-dimensional submanifold of 5-dimensional Minkowski spacetime given by

$$AdS = \{(x^0, \boldsymbol{x}, w) \in \mathbb{R}^5 : (x^0)^2 - |\boldsymbol{x}|^2 + w^2 = 1\}$$

and equipped with the metric inherited by restriction of the ambient 5-dimensional Minkowski metric to the submanifold.

(e) The "right wedge" region in Minkowski spacetime

$$W_R = \{(x^0, \boldsymbol{x}) \in \mathbb{R}^4 : x^1 > 0,\ \ |x^0| < x^1\}$$

(f) The "timelike slab" in $1+d$-dimensional Minkowski spacetime:

$$\mathsf{T} = \{(x^0, x^1, \ldots, x^d) : -s < x^1 < s\}$$

for some positive number s.

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(B)

(i) Show that in Minkowski spacetime, every pair of distinct points can be connected by a smooth, spacelike curve.

(ii) Show that there cannot be closed timelike curves in Minkowski spacetime.

(iii) More generally, use the equivalent characterizations of global hyperbolicity to show that a globally hyperbolic spacetime cannot contain any closed timelike curve.

(iv) Show that in the "lying clinder spacetime" $S^1 \times \mathbb{R}$, where the time-axis has been rolled up into S^1, every pair of distinct points can be connected by a smooth, timelike curve.

Problem 3.2 [3 points]

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid rotation in 3-dim. Euclidean space, described by

$$\phi(x)^j = R^j{}_k x^k \,, \quad x = (x^k)_{k=1,2,3}$$

where $(R^j{}_k)$ is a real orthogonal 3×3 matrix with determinant equal to 1.

In the standard Cartesian coordinates for $\mathbb{R}^3$ chosen, derive the coordinate expression for the action of the pull-back $\phi_\#$ on $\binom{r}{s}$ tensor fields on $\mathbb{R}^3$.

Do the same for the case that $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is an arbitrary diffeomorphism.

Problem 3.3 [2 points]

Let the 2-dimensional manifold $M = \{y = (y^1, y^2, y^3) \in \mathbb{R}^3 : (y^2)^2 + (y^3)^2 = 1\} \simeq \mathbb{R} \times S^1$ be equipped with a coordinate chart having coordinates $x^0 \in \mathbb{R}$, $x^1 \in (0, 2\pi)$, implicitly defined by

$$y(x^0, x^1) = \begin{pmatrix} x^0 \\ \cos(x^1) \\ \sin(x^1) \end{pmatrix} .$$

Let a Lorentzian metric g_{ab} be defined on M as the C^∞ extension of the coordinate expression

$$\begin{pmatrix} -\mathrm{e}^{-|x^0|^2} & 0 \\ 0 & \sin(x^1/2)^2 + 1 \end{pmatrix}$$

For any continuous function $f : M \to \mathbb{R}$, give an as explicit as possible (formal) expression for the integral

$$\int_M f \, d|\mathrm{vol}_g|$$

using the coordinates (x^μ). Show that, if f is polynomially bounded in these coordinates, i.e. $|f(y(x^0, x^1))| \leq C(1+|(x^0, x^1)|)^N$ for suitable numbers C and N, then the integral is well-defined.