1st Postulate

To every type of quantum mechanical system one assigns a Hilbert space $\mathcal{H}$ (the system's Hilbert space)

- Typical (elementary) quantum mechanical systems:
  Atoms, electrons, elementary particles, photons;
  degress of freedom of such systems, e.g. spin-orientation, polarization;
  collective excitations, "quasi-particles"
- Type: Systems of different type can be distinguished,
  e.g.: charge, mass
  Systems of different type cannot be superposed.

2nd Postulate

The **procedures of preparation** of ensembles of identically prepared quantum mechanical systems correspond to the **states** of the system.

Formal description

- A state is a linear functional (positive and normalized)

$$\omega : \mathsf{B}(\mathcal{H}) \to \mathbb{C}$$

  of the form

$$\omega(A) = \mathrm{Tr}(\varrho A) \quad (A \in \mathsf{B}(\mathcal{H}))$$

  where ϱ is a **density matrix**

- Special case:

$$\varrho = |\psi\rangle\langle\psi| \quad \text{1-dim projector}$$

$$\Rightarrow \quad \mathrm{Tr}(\varrho A) = (\psi, A\psi)$$

3rd Postulate

The **observables = procedures of measurement** correspond to the selfadjoint (s.a.) operators A in $\mathcal{H}$

If ω is a state and A is an observable, then

$$\omega(A) = \mathrm{Tr}(\varrho A)$$

is the **expectation value of A in the state** ω:
The **statistical mean** over results of measurements of the measurement procedure represented by A, measured on an ensemble of identically prepared systems, represented by ω. (Ensemble has to be large, ideally: ∞)

4th Postulate

To every system is also assigned a **Hamilton operator**, an s.a. in $\mathcal{H}$, determining the **time evolution** of the system through its unitary group U_t^H, $t \in \mathbb{R}$,

$$U_t^H = \mathrm{e}^{itH/\hbar}$$

If $\omega(A) = \mathrm{Tr}(\varrho A)$ is the expectation value of observable A in state ω at some time t_0, then

$$\omega(U_t^H A (U_t^H)^*) = \mathrm{Tr}(\varrho U_t^H A (U_t^H)^*) = \mathrm{Tr}((U_t^H)^* \varrho U_t^H A)$$

is the expectation value of the same observable at time $t_0 + t$.

(1) $\mathrm{B}(\mathcal{H})$ = algebra of bounded linear operators on $\mathcal{H}$

$$A \in \mathrm{B}(\mathcal{H}) \quad \Leftrightarrow \quad A : \mathcal{H} \to \mathcal{H} \ \text{ is linear, and}$$
$$||A|| = \sup_{||\psi||=1} ||A\psi|| < \infty\,, \quad \text{where}$$
$$||\psi|| = \sqrt{(\psi,\psi)} \quad (\psi \in \mathcal{H})$$

(2) $X \in \mathrm{B}(\mathcal{H})$ is a **trace-class operator** if:
For all $A, B \in \mathrm{B}(\mathcal{H})$, for any ONB $\{\psi_j\}_{j=1}^{\dim(\mathcal{H})}$:

$$\sum_{j=1}^{\dim(\mathcal{H})} (\psi_j, AXB\psi_j) \quad \text{is absolutely convergent}$$

$$\Rightarrow \ \mathrm{Tr}(X) = \sum_{j=1}^{\dim(\mathcal{H})} (\psi_j, X\psi_j) \quad \text{independent of chosen ONB}$$

Properties of the **trace** Tr

- X trace-class $\Rightarrow$ AXB trace-class $(A, B \in \mathrm{B}(\mathcal{H}))$
- $\mathrm{Tr}(AXB) = \mathrm{Tr}(XBA) = \mathrm{Tr}(BAX)$ **cyclicity**
- X trace-class $\Rightarrow$ X^* trace-class $[\ (\psi, X\phi) = (X^*\psi, \phi)\ ]$
- $\mathrm{Tr}(X^*) = \overline{\mathrm{Tr}(X)}$
- $\mathrm{Tr}(aX + bY) = a\mathrm{Tr}(X) + b\mathrm{Tr}(Y)$ $(a, b \in \mathbb{C},\ X, Y$ trace-class$)$
- Any hermitean trace-class operator $X = X^*$ has a **spectral representation**:

$$X = \sum_{j=1}^{\dim(\mathcal{H})} x_j |\chi_j\rangle\langle\chi_j| \text{ with } x_j \in \mathbb{R},\ \{\chi_j\} \text{ ONB}$$

$$X\chi_j = x_j\chi_j \quad [\chi_j \text{ eigenvectors with eigenvalues } x_j]$$

$$\sum_j |x_j| < \infty$$

A hermitean trace-class operator ϱ is a **density matrix** if its spectral decomposition

$$\varrho = \sum_{j=1}^{\dim(\mathcal{H})} r_j |\psi\rangle\langle\psi|$$

has the properties

- $r_j \geq 0 \quad \Leftrightarrow \quad \mathrm{Tr}(\varrho A^* A) \geq 0 \quad (A \in \mathrm{B}(\mathcal{H}))$
- $\sum_j r_j = 1 \quad \Leftrightarrow \quad \mathrm{Tr}(\varrho) = 1$

$$\{ \text{ states } \omega \} = \{ \text{ density matrices } \varrho \}$$

(3)
- $A \in \mathrm{B}(\mathcal{H})$ is **selfadjoint** (hermitean) if $A^* = A$
- $U \in \mathrm{B}(\mathcal{H})$ is **unitary** if $U^* U = \mathbf{1} = UU^*$
- $P \in \mathrm{B}(\mathcal{H})$ is a **projector** if $P^* = P$ and $P^2 (= PP) = P$
- A **1-dim projector** $|\psi\rangle\langle\psi|$ is given by a unit vector ψ:

$$|\psi\rangle\langle\psi|\chi = (\psi, \chi)\psi \qquad (||\psi|| = 1)$$

(4) In quantum mechanics, observables are often **unbounded** s.a. or essentially s.a. operators (ess.s.a.) — in this case, the definition of self-adjointness is more complicated.

(4a) Unbounded operators are not defined on all of $\mathcal{H}$ but only on a dense subspace:

$$A : D(A) \to \mathcal{H}\,, \quad D(A) = \text{ domain of } A$$

They are linear operators:

$$A(a\psi + b\chi) = aA\psi + bA\chi\,, \quad (a, b \in \mathbb{C},\ \psi, \chi \in D(A))$$

If A is self-adjoint depends strongly on choice of $D(A)$. Therefore, precise notation for an unbounded operator is

$$(A, D(A)) \quad \text{notation includes domain of } A$$

An operator $(A, D(A))$ is **unbounded** if there is a sequence

$$\psi_n \in D(A) \;\; (n \in \mathbb{N}) \;\; \text{with} \;\; ||\psi_n|| = 1 \;\; \text{and} \;\; ||A\psi_n|| \to \infty \; (n \to \infty)$$

(4b) Let $(A, D(A))$ be an unbounded operator in $\mathcal{H}$.
An operator $(B, D(B))$ is called an **extension** of $(A, D(A))$ if

$$D(A) \subset D(B) \;\; \text{and} \;\; B\psi = A\psi \;\; \text{for all } \psi \in D(A)$$

$$\Leftrightarrow \qquad B|_{D(A)} = A$$

(4c) To an unbounded operator $(A, D(A))$ one can define the **adjoint operator** $(A^*, D(A^*))$ through the following conditions:

- $(\psi, A\chi) = (A^*\psi, \chi)$ for all $\psi \in D(A^*)\,,\ \chi \in D(A)$
- There is no proper extension of $(A^*, D(A^*))$ fulfilling the previous condition

The domain $D(A^*)$ must be dense. This is the case if $(A, D(A))$ is bounded, or if $(A, D(A))$ is **symmetric**

(4d) $(A, D(A))$ is **symmetric** (or **hermitean**) if

$$(A\psi, \chi) = (\psi, A\chi) \quad \text{for all } \psi, \chi \in D(A)$$

(4e) $(A, D(A))$ is **selfadjoint** if

$$D(A^*) = D(A) \quad \text{and} \quad A^* = A$$
$$\Leftrightarrow \ (A^*, D(A^*)) = (A, D(A))$$

Every selfadjoint operator is hermitean, but the converse does not hold.

There are examples of hermitean operators which are not selfadjoint, or do not even possess any selfadjoint extension.

This happens if $D(A)$ is a proper subset of $D(A^*)$

Possibilities for a hermitean operator which is **not** selfadjoint

(4f) $(A, D(A))$ is **essentially selfadjoint** if
there is a **unique** selfadjoint extension $(\bar{A}, D(\bar{A}))$ of $(A, D(A))$

(4g) $(A, D(A))$ is **selfadjoint extendable** if
there exist several (different) selfadjoint extensions of $(A, D(A))$

(4h) $(A, D(A))$ is **not selfadjoint extendable** if
there are no selfadjoint extensions of $(A, D(A))$

(5) In QM typically for ∞-dim system Hilbert spaces:

- $\mathcal{H} = L^2(G \subset \mathbb{R}^n)$
- unbounded observables are (partial) differential operators
- the **choice of a specific domain** corresponds to **boundary conditions**

(6) **Unitary groups**

A family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators is a **continuous unitary group** if

$$U_t U_s = U_{t+s}\,, \quad U_{t=0} = \mathbf{1}\,, \quad U_{-t} = (U_t)^*$$
$$||U_t\psi - \psi|| \to 0 \ \text{ for } \ t \to 0 \quad (\psi \in \mathcal{H})$$

A continuous unitary group $\{U_t\}_{t\in\mathbb{R}}$ defines **uniquely** an s.a. operator $(A, D(A))$ by

$$(\star) \quad D(A) = \{\psi \in \mathcal{H} : \lim_{t\to 0} \frac{1}{t}(U_t\psi - \psi) \text{ exists } \}$$

$$(\star\star) \quad A\psi = \frac{1}{i}\frac{d}{dt}\bigg|_{t=0} U_t\psi \quad (\psi \in D(A))$$

Conversely, for every s.a. operator $(A, D(A))$ there is a **unique** continuous unitary group $\{U_t\}_{t\in\mathbb{R}}$ fulfilling $(\star)$ and $(\star\star)$

$(A, D(A))$ is the **generator** of $\{U_t\}_{t\in\mathbb{R}}$

$\{U_t\}_{t\in\mathbb{R}}$ is the **unitary group of** $(A, D(A))$

(7) Let $(A, D(A))$ be hermitean.

$a \in \mathbb{R}$ is an **eigenvalue** of $(A, D(A))$ if there is $\psi \in D(A)$, $\psi \neq 0$ with

$$A\psi = a\psi \,,$$

then ψ is an **eigenvector** of $(A, D(A))$.

- Eigenvectors with different eigenvalues are orthogonal
- There can be orthogonal eigenvectors with the same eigenvalue. Then the eigenvalue is **degenerate**
- $\mathcal{N}(A, a) =$ subspace of $\mathcal{H}$ spanned by the eigenvectors with eigenvalue a

(8)

More general than eigenvalues: spectral values

Let $(A, D(A))$ be essentially selfadjoint.

a is a **spectral value** of $(A, D(A))$ if there is a sequence $\psi_n \in D(A)$ with

$$||\psi_n|| = 1 \qquad \text{and} \qquad ||(a\mathbf{1} - A)\psi_n|| \to 0 \;\; (n \to \infty)$$

$\mathrm{spec}(A)$ = **spectrum of** A = set of all spectral values of $(A, D(A))$

$\mathrm{spec}(A)$ is a closed subset of $\mathbb{R}$

(9)

For every s.a. $(A, D(A))$ there is a unique **projector valued measure (spectral measure)**

$$P = P^A : \{\text{measurable subsets of } \mathbb{R}\} \to \{\text{projectors on } \mathcal{H}\}$$

- $J \cap I = \emptyset \quad \Rightarrow \quad P(J)P(I) = 0$
- $P(\bigcup_n J_n) = \sum_n P(J_n) \quad \text{if} \quad J_k \cap J_\ell = \emptyset$ for $k \neq \ell$
- $P(\emptyset) = 0\,, \qquad P(\mathbb{R}) = \mathbf{1}$
- $P^A(J) = 0$ if $J \cap \mathrm{spec}(A) = \emptyset$

Spectral measure P^A can be used to define functions of A

char_J characteristic function of J

step-function $h(a) = \sum_{k=1}^{N} r_k \mathrm{char}_{J_k}$, $J_1, \ldots, J_N$ pairw. disjoint intervals

define $h(A) = \sum_{k=1}^{N} r_k P^A(J_k)$

h_n sequence of step-functions with $h_n(a) \to f(a)$ $(n \to \infty)$

define $f(A) = \lim_{n\to\infty} h_n(A) = \int f(a)\, dP^A(a)$

Properties of $f(A)$

- f bounded $\quad\Rightarrow\quad f(A)$ bounded
- $A^n = \int a^n \, dP^A(a)$
- $U_t^A = \int \mathrm{e}^{ita} \, dP^A(a)$
- $f(A) = (2\pi)^{-1} \int \widehat{f}(t) \, U_t^A \, dt \,, \quad \widehat{f} =$ Fourier-transform of f

Consequence of 3rd Postulate

$$\omega(P^A(J)) = \mathrm{Tr}(\varrho P^A(J))$$

is the **probability of finding measurement values of the observable** A **in the interval** J if the system is in state ω (given by density matrix ϱ)
** is equal to 0 if $J \cap \mathrm{spec}(A) = \emptyset$ **

II. Groups and Hilbert space representations

- Continuous groups are typically **Lie groups**:

  finite dimensional differentiable manifolds equipped with a group multiplication which is compatible with the manifold structure

- Here: Our main concern will be the **Lorentz group** and the **Poincaré group**, i.e. the **spacetime symmetry groups** of **Minkowski spacetime** (mostly 4-dimensional)

  These groups, resp. suitable subgroups, belong to the class of **continuous matrix groups**

  In QFT, there appear also **internal symmetries** associated with **charges**. In many cases (e.g. those relevant for the standard model) these are also described by continuous matrix groups.

- A **continuous matrix group** is a

  continuous subgroup G of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$
  (the group of real or complex $n \times n$ matrices)

  - ⋆ group multiplication is **matrix multiplication**
  - ⋆ unit element is the **unit matrix 1**

II. Groups and Hilbert space representations

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  - ⋆ group multiplication is **matrix multiplication**
  - ⋆ unit element is the **unit matrix 1**

- Natural concept of **continuity**:

$$g' = (g'_{jk}) \longrightarrow (g_{jk}) = g$$

(convergence in G) if

$$g'_{jk} \longrightarrow g_{jk}$$

(convergence of all matrix elements)

- Natural concept of **differentiability**:

  A map

$$\underline{g} : \mathbb{R}^m \to G \,, \quad x \mapsto \underline{g}(x) = (\underline{g}_{jk}(x)) \quad \text{is} \quad C^N \quad (N \in \mathbb{N}_0)$$

  if all

$$x \mapsto \underline{g}_{jk}(x) \quad \text{are} \quad C^N$$

II. Groups and Hilbert space representations

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  if all

  $$x \mapsto \underline{g}_{jk}(x) \quad \text{are} \quad C^N$$

- Concept of **open neighbourhoods** of points $g \in G$ is inherited by embedding G into $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$

- Sufficiently small neighbourhoods N of any $g \in G$
  can be "traced out" by families $\{\underline{g}_\alpha\}$ of differentiable curves

  $$\underline{g}_\alpha : (-\epsilon_g, \epsilon_g) \to G\,, \quad t \mapsto \underline{g}_\alpha(t)$$

  where $\underline{g}_\alpha(0) = g$ and
  different curves intersect only in g.

  (This relates to the concept of the **exponential map** of a Lie group)

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(This relates to the concept of the **exponential map** of a Lie group)

- **Formalization of these concepts leads to the rigorous mathematical definition of continuous matrix groups as special subclass of finite-dimensional Lie groups**

Some examples —

- $GL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det(A) \neq 0\}$
- $GL(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : \det(A) \neq 0\}$
- n-dimensional **orthogonal group**:
  $O(n) = \{D \in GL(n, \mathbb{R}) : D^T D = \mathbf{1}\}$
- n-dimensional **special orthogonal group**:
  $SO(n) = \{R \in GL(n, \mathbb{R}) : R^T R = \mathbf{1} \text{ and } \det(R) = 1\}$
- n-dimensional **unimodular group**:
  $U(n) = \{U \in GL(n, \mathbb{C}) : U^* U = UU^* = \mathbf{1}\}$
- n-dimensional **special unimodular group**:
  $SU(n) = \{W \in GL(n, \mathbb{C}) : W^* W = WW^* = \mathbf{1} \text{ and } \det(W) = 1\}$

More examples

- n-dimensional **Lorentz group**:
  $O(1, n-1) = \{\Lambda \in GL(n, \mathbb{R}) : \Lambda^T \boldsymbol{\eta} \Lambda = \boldsymbol{\eta}\}$,
  $\boldsymbol{\eta} = \mathrm{diag}(1, -1, \ldots, -1)$
- n-dimensional **special, orthochronous Lorentz group**:
  $SO_+(1, n-1) = \{\Lambda \in O(1, n-1) : \det(\Lambda) = 1 \text{ and } \Lambda_{00} > 0\}$
- $2n$-dimensional **(real) symplectic group**:
  $Sp(2n) = \{S \in GL(2n, \mathbb{R}) : S^T J S = J\}$,

$$J = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

- n-dimensional **upper triangular group**
  $T(n) = \{F \in GL(n, \mathbb{C}) : F_{jk} = 0 \text{ if } j > k\}$

Connectedness

Definition (A)

Let G be a continuous matrix group.

G is called **connected** if

for any pair g_0, g_1 of elements in G there is a continuous curve

$$\underline{g} : [0,1] \to G\,, \qquad t \mapsto \underline{g}(t)$$

so that $\quad \underline{g}(0) = g_0 \quad$ and $\quad \underline{g}(1) = g_1\;.$

Connectedness

Example

$O(3) = \{D \in GL(3, \mathbb{R}) : D^T D = \mathbf{1}\}$

"Generalized rotations", including also **reflections**, e.g. $-\mathbf{1}$

$O(3)$ is **not connected**

To see this:

$g_0 = \mathbf{1}$ and $g_1 = -\mathbf{1}$ are in $O(3)$

$\det(g_0) = 1$ and $\det(g_1) = -1$

If $\underline{g}$ was a continuous curve in $O(3)$ connecting g_0 and g_1, then:

- $\det(\underline{g}(t))$ is continuous in t
- $\det(\underline{g}(0)) = 1$, but $\det(\underline{g}(1)) = -1$
- $\Rightarrow \det(\underline{g}(t)) = 0$ for some t
- $\Rightarrow$ *contradiction* since $\underline{g}(t) \in GL(3, \mathbb{R})$ for all t
- $\Rightarrow$ $O(3)$ is **not connected**

Connectedness

Example

$SO(3) = \{R \in GL(3, \mathbb{R}) : R^T R = \mathbf{1}, \ \det(R) = 1\}$

"Proper rotations" without reflections

$SO(3)$ is **connected**

Any element D in $O(3)$ has either $\det(D) = 1$ or $\det(D) = -1$

$SO(3)$ is the part of $O(3)$ where all elements have determinant $= 1$

This part of $O(3)$ is **connected**, **contains 1**
and is a **subgroup of** $O(3)$

$SO(3)$ is the **unit connected component** of $O(3)$.

Simple connectedness

Definition (B)

Let G be a continuous matrix group.

G is called **simply connected** if

any closed continuous curve in G can be continuously contracted to any of its points.

Simple connectedness

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Let G be a continuous matrix group.

G is called **simply connected** if

any closed continuous curve in G can be continuously contracted to any of its points.

- $\underline{g} : [0,1] \to G$, continuous, is **closed** if $\underline{g}(0) = \underline{g}(1)$
- $\underline{F} : [0,1] \times [0,1] \to G$ is a **continuous contraction** of $\underline{g}$ to $g = \underline{g}(0)$ if

(i) $\underline{F}$ is continuous

(ii) $\underline{F}(1,t) = \underline{g}(t)$ for all $t \in [0,1]$

(iii) $\underline{F}(\mu,1) = \underline{F}(\mu,0) = g$ for all $\mu \in [0,1]$

(iv) $\underline{F}(0,t) = g$

Simple connectedness

Example

$U(1) = \{\mathrm{e}^{ir} : r \in \mathbb{R}\}$

$U(1)$ is connected, but **not simply** connected

The curve

$$\underline{g}(t) = \mathrm{e}^{i2\pi t}\,, \quad t \in [0,1]$$

is a closed continuous curve in $U(1)$

but $\underline{g}$ is a like a **closed loop around a rod**

and cannot be continuously contracted to any of its points.

Isomorphy

Let G_1 and G_2 be two continuous matrix groups.
They are called **isomorphic** if there is

$$\phi : G_1 \to G_2\,, \quad \text{bijective}, \;\; C^\infty\,, \quad \text{with}$$

$$\phi(gg') = \phi(g)\phi(g')\,, \quad \phi(\mathbf{1}_{(1)}) = \mathbf{1}_{(2)}$$

Isomorphy

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- They are called **locally isomorphic** if there are

$$\text{open neighbourhoods} \quad N_1 \quad \text{of} \quad \mathbf{1}_{(1)} \quad \text{and} \quad N_2 \quad \text{of} \quad \mathbf{1}_{(2)}\,,$$

$$\phi : N_1 \to N_2\,, \quad \text{bijective,} \quad C^\infty\,, \quad \text{with}$$

$$\phi(gg') = \phi(g)\phi(g')\,, \quad \phi(\mathbf{1}_{(1)}) = \mathbf{1}_{(2)}$$

$$\text{for all} \quad g,\, g' \in N_1 \quad \text{with} \quad gg' \in N_1$$

Lie algebra

Let G be a continuous matrix group in $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.

- $\boldsymbol{x} \in M_{n\times n}(\mathbb{R})$ or $M_{n\times n}(\mathbb{C})$
  is called **tangent** to G (at $\mathbf{1}$) if there is

$$\underline{g} : (-\epsilon, \epsilon) \to G\,, \quad t \mapsto \underline{g}(t)\,, \quad C^1\,, \quad \text{with}$$

$$\underline{g}(0) = \mathbf{1} \quad \text{and} \quad \boldsymbol{x} = \dot{\underline{g}}(0) = \left.\frac{d}{dt}\,\underline{g}(t)\right|_{t=0}$$

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$$\underline{g}(0) = \mathbf{1} \quad \text{and} \quad \boldsymbol{x} = \dot{\underline{g}}(0) = \frac{d}{dt}\,\underline{g}(t)\Big|_{t=0}$$

- $$\mathfrak{G} = \{\boldsymbol{x} : \boldsymbol{x} \ \text{ is tangent to } \ G\}$$

  is the **Lie algebra of** G, $\quad \mathfrak{G} = \mathrm{Lie}(G)$

(C) Lie algebra: Properties

(i) $\mathfrak{G}$ is a **real-linear vector space**:

$$\boldsymbol{x}\,,\ \boldsymbol{y} \in \mathfrak{G}\,, \quad \lambda, \mu \in \mathbb{R} \quad \Rightarrow \quad \lambda\boldsymbol{x} + \mu\boldsymbol{y} \in \mathfrak{G}$$

(ii)

$$\boldsymbol{y} \in \mathfrak{G}\,, \quad g \in G \quad \Rightarrow \quad g\boldsymbol{y}g^{-1} \in \mathfrak{G}$$

and

$$\boldsymbol{x}\,,\ \boldsymbol{y} \in \mathfrak{G}\,, \quad \Rightarrow \quad [\boldsymbol{x}, \boldsymbol{y}] = \boldsymbol{xy} - \boldsymbol{yx} \in \mathfrak{G}$$

N.B. Note that in general, neither $\boldsymbol{xy}$ nor $g\boldsymbol{y}$ or $\boldsymbol{y}g$ will be in $\mathfrak{G}$!

(C) Lie algebra: Properties — Sketch of proof

(i)

- Let $\boldsymbol{x} = \dot{\underline{g}}(0)$ be in $\mathfrak{G}$
- rescaled curve $\underline{h}(t) = \underline{g}(\lambda t)$ also fulfills $\underline{h}(0) = \mathbf{1}$
- $\dot{\underline{h}}(0) = \lambda \dot{\underline{g}}(0) = \lambda \boldsymbol{x}$

$\Rightarrow$ $\lambda \boldsymbol{x}$ is in $\mathfrak{G}$

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- $\dot{\underline{h}}(0) = \lambda \dot{\underline{g}}(0) = \lambda \boldsymbol{x}$

$\Rightarrow$ $\lambda \boldsymbol{x}$ is in $\mathfrak{G}$

- Let $\boldsymbol{x} = \dot{\underline{g}}(0)$ and $\boldsymbol{y} = \dot{\underline{h}}(0)$ be in $\mathfrak{G}$
- $\underline{k}(t) = \underline{g}(t)\underline{h}(t)$ is differentiable curve in G with $\underline{k}(0) = \mathbf{1}$
- $\dot{\underline{k}}(0) = \dot{\underline{g}}(0)\underline{h}(0) + \underline{g}(0)\dot{\underline{h}}(0) = \boldsymbol{x} + \boldsymbol{y}$

$\Rightarrow$ $\boldsymbol{x} + \boldsymbol{y}$ is in $\mathfrak{G}$

(C) Lie algebra: Properties — Sketch of proof

(ii)

- Let $\boldsymbol{y} = \dot{\underline{h}}(0)$ be in $\mathfrak{G}$ and let $g \in G$
- then $\underline{k}(t) = g\underline{h}(t)g^{-1}$ is differentiable curve in G with $\underline{k}(0) = \mathbf{1}$

$\Rightarrow$ $g\boldsymbol{y}g^{-1} = g\dot{\underline{h}}(0)g^{-1} = \dot{\underline{k}}(0)$ is in $\mathfrak{G}$

(C) Lie algebra: Properties — Sketch of proof

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- Let $\boldsymbol{y} = \dot{\underline{h}}(0)$ be in $\mathfrak{G}$ and let $g \in G$
- then $\underline{k}(t) = g\underline{h}(t)g^{-1}$ is differentiable curve in G with $\underline{k}(0) = \mathbf{1}$

$\Rightarrow$ $g\boldsymbol{y}g^{-1} = g\dot{\underline{h}}(0)g^{-1} = \dot{\underline{k}}(0)$ is in $\mathfrak{G}$

- Let $\boldsymbol{x} = \dot{\underline{g}}(0)$ be in $\mathfrak{G}$
- then

$$\boldsymbol{z}_t = \frac{1}{t}(\underline{g}(t)\boldsymbol{y}\underline{g}^{-1}(t) - \boldsymbol{y}) \quad \text{is in} \quad \mathfrak{G} \quad \text{for all} \quad t$$

$\Rightarrow$ $\boldsymbol{xy} - \boldsymbol{yx} = \lim_{t\to 0} \boldsymbol{z}_t$ is in $\mathfrak{G}$

Lie algebra: Further properties

The commutator bracket $[\boldsymbol{x}, \boldsymbol{y}]$ is a **natural algebraic structure** of the Lie-algebra of any continuous matrix group.

Definition (D)

Two Lie-algebras $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are **isomorphic**
if there is

$$\Phi : \mathfrak{G}_1 \to \mathfrak{G}_2 \quad \text{linear, bijective,} \quad \text{with}$$

$$[\Phi(\boldsymbol{x}), \Phi(\boldsymbol{y})]_{(2)} = \Phi([\boldsymbol{x}, \boldsymbol{y}]_{(1)})$$

Remark: Structure constants

Let $\mathfrak{G}$ be a Lie-algebra

- $\boldsymbol{b}_1 \ldots, \boldsymbol{b}_N$ a vector space basis of $\mathfrak{G}$
- Commutator bracket (or Lie-bracket) can be expanded in terms of the basis:
$$[\boldsymbol{b}_j, \boldsymbol{b}_k] = \sum_{\ell=1}^{N} f_{jk}^{\ell} \boldsymbol{b}_\ell$$
- The f_{jk}^{ℓ} are called **structure constants** of $\mathfrak{G}$, they are unique up to basis transformations: Let $\boldsymbol{b}'_r = \sum_{s=1}^{N} M_{rs} \boldsymbol{b}_s$ be another basis, then
$$f'^{r}_{pq} = \sum_{j,k,\ell} M_{pj} M_{qk} (M^{-1})^{\ell r} f_{kj}^{\ell}\,, \quad (M^{-1})^{\ell r} = (M^{-1})_{\ell r}$$

Remark: Structure constants

Thus, **up to basis transformations, the structure constants are unique** and characteristic for any Lie-algebra.

Any two Lie-algebras having structure constants which are equal up to basis transformations are isomorphic.

Theorem (E)

Let G_1 and G_2 be two continuous matrix groups,
$\mathfrak{G}_1$ and $\mathfrak{G}_2$ their Lie-algebras.

Then

G_1 and G_2 are **locally isomorphic** if and only if

$\mathfrak{G}_1$ and $\mathfrak{G}_2$ are **isomorphic**.

Theorem (F) — Universal Covering Group

Let G be a **connected** continuous matrix group.
Then there is a Lie-group $\widetilde{G}$ (in many cases, this is again a continuous matrix group) which is

- ⋆ **connected**
- ⋆ **simply connected**
- ⋆ **locally isomorphic** to G

$\widetilde{G}$ is unique up to isomorphism and called **universal covering group** of G.

Theorem (F) — Universal Covering Group

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Then there is a Lie-group $\widetilde{G}$ (in many cases, this is again a continuous matrix group) which is

connected, **simply connected** and **locally isomorphic** to G

$\widetilde{G}$ is unique up to isomorphism and called **universal covering group** of G.

The construction of $\widetilde{G}$ from G brings about a canonical map

$$\overline{\phi} : \widetilde{G} \to G$$

which preserves the group structure and is **surjective**, but only **locally injective** — on a neighbourhood of $\widetilde{\mathbf{1}}$.

Restriction of $\overline{\phi}$ to such neighbourhood is a **local isomorphism**

$\overline{\phi}$ is the **covering map** or **covering projection**

Example: $\mathrm{Lie}(SU(n))$

$$G = SU(n) = \{U \in GL(n,\mathbb{C}) : U^* U = \mathbf{1}\,, \quad \det(U) = 1\}$$

$$\mathrm{Lie}(SU(n)) = \mathfrak{SU}(2),$$

$$\mathfrak{SU}(n) = \{A \in M_{n\times n}(\mathbb{C}) : A^* = -A\,, \quad \mathrm{trace}(A) = 0\}$$

For $A \in \mathfrak{SU}(n)$,

$U_t = \mathrm{e}^{tA} \quad (t \in \mathbb{R})$ is unitary and has determinant $= 1$

(matrix exponential in $M_{n\times n}(\mathbb{C})$)

$$\Rightarrow \quad U_t \in SU(n)\,, \quad t \mapsto U_t \text{ is } C^1\,, \quad U_{t=0} = \mathbf{1}$$

$$\frac{d}{dt} U_t \Big|_{t=0} = A$$

Example: $\mathrm{Lie}(SU(2))$

A **basis** of $\mathfrak{SU}(2)$ is related to the Pauli matrices:

$$\boldsymbol{s}_1 = \frac{i}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{s}_2 = \frac{i}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\boldsymbol{s}_3 = \frac{i}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

The commutator bracket (Lie-bracket) between the basis elements is

$$[\boldsymbol{s}_j, \boldsymbol{s}_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell}\boldsymbol{s}_\ell$$

$\epsilon_{jk\ell}$ is the totally antisymmetric unit tensor

Example: $\mathrm{Lie}(SO(n)) = \mathfrak{SO}(n)$

$SO(n) = \{R \in GL(n,\mathbb{R}) : R^T R = \mathbf{1}\,, \quad \det(R) = 1\}$

$\mathfrak{SO}(n) = \{L \in M_{n\times n}(\mathbb{R}) : L^T = -L\,, \quad \mathrm{trace}(L) = 0\}$

For all $L \in \mathfrak{SO}(n)$,

$R_\theta = \mathrm{e}^{\theta L}$ is orthogonal and has determinant $= 1$

(matrix exponential in $M_{n\times n}(\mathbb{R})$)

$$\Rightarrow \quad R_\theta \in SO(n)\,, \quad \theta \mapsto R_\theta \text{ is } C^1\,, \quad R_{\theta=0} = \mathbf{1}$$

$$L = \left.\frac{d}{d\theta} R_\theta \right|_{\theta=0}$$

Example: $\mathrm{Lie}(SO(3))$

$R(\vec{n}, \theta) =$ rotation around axis $\vec{n}$ (3-dim real unit vector) by angle θ,

$\vec{e}_1, \vec{e}_2, \vec{e}_3 =$ standard basis vectors of $\mathbb{R}^3$

$$L_j = \frac{d}{d\theta} R(\vec{e}_j, \theta)\Big|_{\theta=0}, \quad \text{then}$$

$L_1\,, \; L_2\,, \; L_3$ form a basis of $\mathfrak{SO}(3)$,

commutator brackets :

$$[L_j, L_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} L_\ell$$

Example: $\mathrm{Lie}(SO(3)) \longleftrightarrow \mathrm{Lie}(SU(2))$

Compare:

$$[L_j, L_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} L_\ell \,, \qquad [\boldsymbol{s}_j, \boldsymbol{s}_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} \boldsymbol{s}_\ell$$

Can construct a **Lie-algebra isomorphism**

$$\Phi : \mathfrak{SO}(3) \to \mathfrak{SU}(2)$$

by setting $\Phi(L_k) = \boldsymbol{s}_k$

and extension by $\mathbb{R}$-linearity

II. Groups and Hilbert space representations — $SU(2) = \widetilde{SO(3)}$

- $\mathfrak{SO}(3) \xrightarrow{\Phi} \mathfrak{SU}(2)$ isomorphic

$\Rightarrow$ $SO(3)$ and $SU(2)$ are **locally isomorphic**

- $SU(2)$ is simply connected
- $SO(3)$ is not simply connected

$\Rightarrow$ $SU(2) = \widetilde{SO(3)}$ (up to isomorphism)

II. Groups and Hilbert space representations — $SU(2) = \widetilde{SO(3)}$

The covering map $\overline{\phi} : SU(2) \to SO(3)$

- $\mathfrak{SU}(2)$ is a real-linear space spanned by $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$ (defined above)
- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 4\mathrm{trace}(\boldsymbol{x}^* \boldsymbol{y})$ is a scalar product on $\mathfrak{SU}(2)$
- $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$ is an ONB for this scalar product
- If $\boldsymbol{v} = \sum_j v_j \boldsymbol{s}_j$, $\quad \boldsymbol{w} = \sum_k w_k \boldsymbol{s}_k$ then

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \vec{v} \bullet \vec{w} = \sum_j v_j w_j\,, \quad \vec{v}\,,\ \vec{w} \in \mathbb{R}^3$$

- Let $U \in SU(2)$. Then $T_U \boldsymbol{v} = U \boldsymbol{v} U^*$ is again in $\mathfrak{SU}(2)$. T_U is linear and since $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ form a basis of $\mathfrak{SU}(2)$,

$$T_U \boldsymbol{v} = \sum_j (R_U \vec{v})_j \boldsymbol{s}_j$$

for some linear map $R_U : \mathbb{R}^3 \to \mathbb{R}^3$.

II. Groups and Hilbert space representations — $SU(2) = \widetilde{SO(3)}$

-

$$\begin{aligned}\langle T_U \boldsymbol{v}, T_U \boldsymbol{w}\rangle &= 4\mathrm{trace}((U\boldsymbol{v}U^*)^* U\boldsymbol{w}U^*)\\ &= 4\mathrm{trace}(U\boldsymbol{v}^* U^* U\boldsymbol{w}U^*)\\ &= 4\mathrm{trace}(\boldsymbol{v}^*\boldsymbol{w})\\ &= \langle \boldsymbol{v}, \boldsymbol{w}\rangle\end{aligned}$$

- This implies

$$\begin{aligned}(R_U\vec{v}) \bullet (R_U\vec{w}) &= \langle T_U \boldsymbol{v}, T_U \boldsymbol{w}\rangle\\ &= \langle \boldsymbol{v}, \boldsymbol{w}\rangle\\ &= \vec{v} \bullet \vec{w}\end{aligned}$$

$\Rightarrow$ R_U is in $O(3)$.

II. Groups and Hilbert space representations — $SU(2) = \widetilde{SO(3)}$

- $SU(2)$ is connected and simply connected (must show separately)
- $$U \mathbf{s}_j U^* = \sum_k (R_U)_{kj} \mathbf{s}_k$$
- Thus, $U \mapsto R_U$ is continuous
- Any $U \in SU(2)$ is continuously connected to $\mathbf{1}$
- Thus, any R_U in the image of $U \mapsto R_U$ is continuously connected to $\mathbf{1}$ $\Rightarrow \quad \det(R_U) = 1 \quad \Rightarrow \quad R_U \in SO(3)$
- For $U \in SU(2) \quad \Rightarrow -U \in SU(2)$
- $R_{(-U)} = R_U$ by eqn above $\Rightarrow \;\; U \mapsto R_U$ is **not injective**
- can check: $U \mapsto R_U$ is surjective

$\Rightarrow$ $SO(3)$ is **not simply connected**,

$$U \mapsto R_U \;\; \text{is the covering map} \;\; SU(2) \xrightarrow{\overline{\phi}} SO(3)$$

Let G be a group (e.g. a continuous matrix group)

Definition (G)

$(U, \mathcal{H})$ is a **unitary representation of** G on the Hilbert space $\mathcal{H}$ if

- $\mathcal{H}$ is a (complex) Hilbert space
- $U : G \to \mathcal{U}(\mathcal{H}) = \{\text{unitary operators on } \mathcal{H}\}$
- $U(gg') = U(g)U(g')\,, \quad U(\mathbf{1}_G) = \mathbf{1}_{\mathcal{U}(\mathcal{H})}\,, \quad U(g^{-1}) = U(g)^*$

A unitary representation $(U, \mathcal{H})$ of G is **continuous** if

$$g_\nu \to g \quad \Rightarrow \quad ||(U(g_\nu) - U(g))\psi|| \to 0 \;\text{ for each }\; \psi \in \mathcal{H}$$

II. Groups and Hilbert space representations — Ray-representations

Let G be a continuous matrix group

Definition (G')

$(U, \mathcal{H})$ is a **unitary ray-representation of** G on the Hilbert space $\mathcal{H}$ if

- $\mathcal{H}$ is a (complex) Hilbert space
- $U : G \to \mathcal{U}(\mathcal{H}) = \{\text{unitary operators on } \mathcal{H}\}$
- $U(gg') = \Omega(g,g')U(g)U(g')$ with $\Omega(g,g') \in U(1)$
- $U(\mathbf{1}_G) = \mathbf{1}_{\mathcal{U}(\mathcal{H})}$

A unitary ray representation $(U, \mathcal{H})$ is **locally continuous** (at $\mathbf{1}_G$) if there is an open neighbourhood N of $\mathbf{1}_G$ so that

$$g \mapsto U(g)\,, \quad g, g' \mapsto \Omega(g,g') \quad (g, g' \in N)$$

are continuous.

Let G be a continuous matrix group which is also a semi-simple Lie-group (e.g. $SO(3)$, but also translation group $\mathbb{R}^n$, or the proper, orthochronous Poincaré group)

Theorem (H)

Let $(U, \mathcal{H})$ be a **unitary ray-representation of** G on the Hilbert space $\mathcal{H}$ which is locally continuous.

Then $(U, \mathcal{H})$ can be **lifted to a continuous unitary representation of the covering group** $\widetilde{G}$ of G. This means:

- There is an open neighbourhood N of $\mathbf{1}_G$ on which $U(g)$ $(g \in N)$ can be re-defined (by making choices of phase) so that $\Omega(g, g') = 1$ $(g, g' \in N)$.
- There is a continuous unitary representation $(\widetilde{U}, \mathcal{H})$ of $\widetilde{G}$ so that

$$\widetilde{U}(\widetilde{g}) = U(\overline{\phi}(\widetilde{g})) \quad (\widetilde{g} \in \widetilde{G})$$

III. Minkowski spacetime, Poincaré group

QFT starts from the attempting to —

- unify the principles of QM and special relativity
- describe elementary particle processes with relativistic energy-momentum transfer (typically: collision processes).

Collision processes of elementary particles then have to be described in the framework of special relativity.

Classical (non-quantum) picture: Paths of particles are worldlines in Minkowski spacetime.

Covariance principle: All inertial observers see the same kind of elementary particles and processes between them — identifyable between different observers in the sense of symmetry transformations in Wigner's sense.

III. Minkowski spacetime, Poincaré group

Spacetime: $\mathcal{M}$ "set of all events", marked by "when" and "where" they happen, i.e. by

- time-coordinate t
- space-coordinates (x^1, x^2, x^3)

with respect to some observer.

A. Einstein 1905:

For all *inertial observers*, light propagation is homogeneous and isotropic and at the same velocity c with respect to their time and space-coordinates (if they use the same clocks and rods):

Light flash ignited at time t_0 and location $\mathbf{x}_0$ reaches at time $t > t_0$ all the space points $\mathbf{x}$ so that

$$c^2(t - t_0)^2 - |\mathbf{x} - \mathbf{x}_0|^2 = 0$$

Then the coordinate transformations between inertial observers are given by **Poincaré transformations**:

$$\begin{pmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix} = \Lambda \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

where:

$$x^0 = ct\,, \quad \tilde{x}^0 = c\tilde{t}$$

The Λ are real 4×4 matrices which fulfill

$$\eta(\Lambda x, \Lambda y) = \eta(x, y) = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

for all

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad y = \begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix}$$

The Λ are the **Lorentz transformations**. They form a group under matrix multiplication, the

$$\text{Lorentz group} \quad \mathscr{L}$$

The

$$a = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

are the **translations**. They form a group under vector addition in $\mathbb{R}^4$, the

$$\text{translation group} \quad \mathscr{T} \simeq \mathbb{R}^4$$

The **Poincaré group** is the **semidirect product** of $\mathscr{L}$ with $\mathscr{T}$,

$$\mathscr{P} = \mathscr{L} \ltimes \mathscr{T}$$

Elements of $\mathscr{L} \ltimes \mathscr{T}$ are pairs (Λ, a), $\Lambda \in \mathscr{L}$, $a \in \mathscr{T}$,
group multiplication law:

$$(\Lambda, a) \circ (\Lambda', a') = (\Lambda\Lambda', \Lambda a' + a)$$

$\mathscr{L}$ consists of **space rotations**, **Lorentz boosts (velocity transformations)**, and **reflections**.
With respect to some inertial observer or **Lorentz frame**:

space rotation

$$\Lambda_R = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & R \end{pmatrix}$$

Lorentz boost in x^1 direction

$$\Lambda_{\mathbf{e}_1}(\theta) = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v/c = \tanh(\theta)$, θ: rapidity

Denoting by $\Lambda_{\mathbf{n}}(\theta)$ the Lorentz boosts along space-direction $\mathbf{n}$, one has:

$$\Lambda_R \Lambda_{\mathbf{e}_1}(\theta) \Lambda_R = \Lambda_{R\mathbf{e}_1}(\theta)$$

$\mathscr{L}$ is not connected. The unit connected component is

$$\mathscr{L}_+ = \{\Lambda \in \mathscr{L} : \det(\mathscr{L}) = 1\} \quad \textbf{proper Lorentz group}$$

The proper Lorentz group has a subgroup

$$\mathscr{L}_+^\uparrow = \{\Lambda \in \mathscr{L}_+ : \Lambda_{00} > 0\} \quad \textbf{proper orthochonous Lorentz group}$$

The transformations in $\mathscr{L}_+^\uparrow$ **preserve space- and time-orientation** and therefore correspond to the transformations between coordinates of physical inertial observers.

Every $\Lambda \in \mathscr{L}_+^\uparrow$ can be **uniquely** written as product of a rotation and a Lorentz boost:

$$\Lambda = \Lambda_{\mathbf{n}}(\theta)\Lambda_R$$

Other Lorentz transformations are obtained from reflections (defined with respect to a given Lorentz frame):

$$\mathtt{T} : \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad \mathtt{P} : \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Then

$$\mathscr{L}_+ = \mathscr{L}_+^\uparrow \cup \mathtt{PT}\mathscr{L}_+^\uparrow, \qquad \mathscr{L} = \mathscr{L}_+ \cup \mathtt{T}\mathscr{L}_+$$

The proper orthochonous Lorentz group $\mathscr{L}_+^\uparrow$ is **not simply connected**.

Theorem (III.A)

- The universal covering group of $\mathscr{L}_+^\uparrow$ is

$$SL(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) : \det(A) = 1\}$$

- The covering map

$$\begin{aligned} SL(2,\mathbb{C}) &\to \mathscr{L}_+^\uparrow \\ A &\mapsto \Lambda(A) \end{aligned}$$

is given by

$$\Lambda(A)_{\mu\nu} = \frac{1}{2}\mathrm{Tr}(A\sigma_\mu A^* \sigma_\nu)$$

where the $\sigma_{1,2,3}$ are the Pauli-matrices and $\sigma_0 = \mathbf{1}_{2\times 2}$

Proof. Exercise problem. It is very similar to the case $SU(2) \to SO(3)$.

In fact, the non-simple connectedness of $\mathscr{L}_+^\uparrow$ originates from this group containing the rotations $SO(3)$.

Proofs are also contained in the books, e.g. Thaller or Sexl-Urbantke.

The irreducible representations of $SL(2,\mathbb{C})$ are given as follows:

- $Ⓢ^k\mathbb{C}^2$ k-fold symmetrized tensor product of $\mathbb{C}^2$
- $V_{k,\ell} = (Ⓢ^k\mathbb{C}^2) \otimes (Ⓢ^\ell\mathbb{C}^2)$
- $D^{(k,\ell)}(A) = (Ⓢ^k A) \otimes (Ⓢ^\ell \overline{A})$
- $(D^{(k,\ell)}, V_{k,\ell})$, where $k,\ell \in \mathbb{N}_0$, are **irreducible, complex-linear representations of $SL(2,\mathbb{C})$.**
- If $k+\ell$ is **even**, then the irrep is called **integer spin**

  if $k+\ell$ is **odd**, then the irrep is called **half-integer spin**