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Quantum Field Theory — Problem Sheet 6

2 pages — Problems 6.1 to 6.4

Problem 6.1

Let $\Phi, \mathcal{H}, \mathcal{D}, \mathscr{U}, \Omega$ be a scalar Wightman quantum field and Θ the PCT operator. Show that for all $(\Lambda, a) \in \mathscr{P}_+^\uparrow$ one has

$$\Theta \mathscr{U}(\Lambda, a)\Theta^{-1} = \mathscr{U}(\Lambda', a')$$

with suitable $(\Lambda', a') \in \mathscr{P}_+^\uparrow$. Determine how (Λ', a') depends on (Λ, a).

Problem 6.2

Let $\mathcal{N} \subset \mathscr{B}(\mathcal{H})$ be a von Neumann algebra, and let U_t $(t \in \mathbb{R})$ be a continuous unitary group on the Hilbert space $\mathcal{H}$ which is assumed to have the property $U_t \mathcal{N} U_t^* = \mathcal{N}$, i.e. α_t $(t \in \mathbb{R})$ defined by $\alpha_t(A) = U_t A U_t^*$ $(A \in \mathcal{N})$ is a one-parametric group of automorphisms of $\mathcal{N}$. Suppose additionally that $\omega(A) = \mathrm{Tr}(\varrho A)$ $(A \in \mathcal{N})$ is a state on $\mathcal{N}$ given by a density matrix ϱ on $\mathcal{H}$.

The state ω is called a *KMS state for* α_t $(t \in \mathbb{R})$ *at inverse temperature* $\beta > 0$ (after Kubo, Martin and Schwinger) if for any pair of operators $A, B \in \mathcal{N}$ there is a function $F_{A,B} : \overline{S_\beta} \to \mathbb{C}$ which is defined and continuous on the closure of the open complex strip

$$S_\beta = \{t + i\eta : t \in \mathbb{R}\,,\ 0 < \eta < \beta\}\,,$$

is analytic on the open strip, and has the boundary values

$$F_{A,B}(t) = \omega(\alpha_t(A)B)\,, \quad F_{AB}(t + i\beta) = \omega(B\alpha_t(A))$$

for all $t \in \mathbb{R}$.

Consider a special case, the Gibbs state: Here, $\mathcal{N} = \mathscr{B}(\mathcal{H})$, $U_t = \mathrm{e}^{itH}$ $(t \in \mathbb{R})$ with some Hamilton operator H which is assumed to be such that $\mathrm{e}^{-\beta H}$ is a trace-class operator on $\mathcal{H}$ for all $\beta > 0$. Show that the Gibbs-state $\omega_\beta(A) = \mathrm{Tr}(\varrho_\beta A)$ is a KMS state at inverse temperature $\beta > 0$ where

$$\varrho_\beta = \frac{1}{\mathrm{Tr}(\mathrm{e}^{-\beta H})}\mathrm{e}^{-\beta H}\,.$$

Hint: To simplify matters, you can assume that $H \geq 0$.

Problem 6.3

For the 2-point function $\mathcal{W}_2$ of the quantized Klein-Gordon field of mass $m > 0$ (cf. Problem 1.2), check explicitly the PCT symmetry property

$$\overline{\mathcal{W}_2(PCT\, f_1 \otimes PCT\, f_2)} = \mathcal{W}_2(f_1 \otimes f_2)\,, \qquad f_j \in \mathscr{S}(\mathbb{R}^4)$$

Problem 6.4

Let $\Phi, \mathcal{H}, \mathcal{D}, \mathscr{U}, \Omega$ be a Wightman quantum field theory. With respect to a chosen Lorentz frame, let H be the global energy operator, i.e. $\mathrm{e}^{itH} = \mathscr{U}(1, te_0)$ where e_0 is the unit-vector of the time-direction.

(1) Show that for any real number $b > 0$ and any vector $\psi \in \mathcal{H}$, the vector $\mathrm{e}^{-bH}\psi$ is an analytic vector for the translations, i.e. the function $a \mapsto \mathscr{U}(1,a)\mathrm{e}^{-bH}\psi$ is analytic.

(2) Let χ, ψ be two unit vectors which are analytic for the translations. Suppose the vectors are found to induce states which coincide on some open spacetime region O (a double cone),

$$(\chi, A\chi) = (\psi, A\psi)$$

for all A in the local algebra $\mathcal{A}(O)$. Conclude that the states are the same: $\chi = \mathrm{e}^{i\phi}\psi$ for some real number ϕ.