Quantum Field Theory — Problem Sheet 6

2 pages — Problems 6.1 to 6.4

Problem 6.1
Let \( \Phi, \mathcal{H}, \mathcal{D}, \mathcal{W}, \Omega \) be a scalar Wightman quantum field and \( \Theta \) the PCT operator. Show that for all \((\Lambda, a)\in \mathcal{P}_+^\uparrow\) one has

\[
\Theta \mathcal{W}(\Lambda, a) \Theta^{-1} = \mathcal{W}(\Lambda', a')
\]

with suitable \((\Lambda', a')\in \mathcal{P}_+^\uparrow\). Determine how \((\Lambda', a')\) depends on \((\Lambda, a)\).

Problem 6.2
Let \( N \subset \mathcal{B}(\mathcal{H}) \) be a von Neumann algebra, and let \( U_t \ (t \in \mathbb{R}) \) be a continuous unitary group on the Hilbert space \( \mathcal{H} \) which is assumed to have the property \( U_t\mathcal{N}U_t^* = \mathcal{N} \), i.e. \( \alpha_t \ (t \in \mathbb{R}) \) defined by \( \alpha_t(A) = U_tAU_t^* \ (A \in \mathcal{N}) \) is a one-parametric group of automorphisms of \( \mathcal{N} \). Suppose additionally that \( \omega(A) = \text{Tr}(\varrho A) \ (A \in \mathcal{N}) \) is a state on \( \mathcal{N} \) given by a density matrix \( \varrho \) on \( \mathcal{H} \).

The state \( \omega \) is called a KMS state for \( \alpha_t \ (t \in \mathbb{R}) \) at inverse temperature \( \beta > 0 \) (after Kubo, Martin and Schwinger) if for any pair of operators \( A, B \in \mathcal{N} \) there is a function \( F_{A,B} : S_\beta \to \mathbb{C} \) which is defined and continuous on the closure of the open complex strip

\[
S_\beta = \{ t + i\eta : t \in \mathbb{R}, \ 0 < \eta < \beta \},
\]

is analytic on the open strip, and has the boundary values

\[
F_{A,B}(t) = \omega(\alpha_t(A)B), \quad F_{AB}(t + i\beta) = \omega(B\alpha_t(A))
\]

for all \( t \in \mathbb{R} \).

Consider a special case, the Gibbs state: Here, \( N = \mathcal{B}(\mathcal{H}), \ U_t = e^{itH} \ (t \in \mathbb{R}) \) with some Hamilton operator \( H \) which is assumed to be such that \( e^{-\beta H} \) is a trace-class operator on \( \mathcal{H} \) for all \( \beta > 0 \). Show that the Gibbs-state \( \omega_\beta(A) = \text{Tr}(\varrho_\beta A) \) is a KMS state at inverse temperature \( \beta > 0 \) where

\[
\varrho_\beta = \frac{1}{\text{Tr}(e^{-\beta H})} e^{-\beta H}.
\]

**Hint:** To simplify matters, you can assume that \( H \geq 0 \).
Problem 6.3
For the 2-point function $W_2$ of the quantized Klein-Gordon field of mass $m > 0$ (cf. Problem 1.2), check explicitly the PCT symmetry property

$$W_2(PCT f_1 \otimes PCT f_2) = W_2(f_1 \otimes f_2), \quad f_j \in \mathcal{S}(\mathbb{R}^4)$$

Problem 6.4
Let $\Phi, \mathcal{H}, \mathcal{D}, \mathcal{W}, \Omega$ be a Wightman quantum field theory. With respect to a chosen Lorentz frame, let $H$ be the global energy operator, i.e. $e^{itH} = \mathcal{U}(1, te_0)$ where $e_0$ is the unit-vector of the time-direction.

1. Show that for any real number $b > 0$ and any vector $\psi \in \mathcal{H}$, the vector $e^{-bH} \psi$ is an analytic vector for the translations, i.e. the function $a \mapsto \mathcal{U}(1, a)e^{-bH} \psi$ is analytic.

2. Let $\chi, \psi$ be two unit vectors which are analytic for the translations. Suppose the vectors are found to induce states which coincide on some open spacetime region $O$ (a double cone),

$$(\chi, A\chi) = (\psi, A\psi)$$

for all $A$ in the local algebra $\mathcal{A}(O)$. Conclude that the states are the same: $\chi = e^{i\phi} \psi$ for some real number $\phi$. 