Problem 4.1
Let $\Phi, \mathcal{H}, \mathcal{D}, \mathcal{W}, \Omega$ be a scalar quantum field fulfilling the Wightman axioms (as discussed in the lectures). The $n$-point functions (Wightman functions) associated with this quantum field are defined by

$$\mathcal{W}_n(f_1 \otimes \cdots \otimes f_n) = (\Omega, \Phi(f_1) \cdots \Phi(f_n)\Omega) \quad (f_j \in \mathcal{S}(\mathbb{R}^4)).$$

Show that the Wightman axioms demanded to hold for the quantum field imply the following properties of the $n$-point functions:

(i) The $\mathcal{W}_n$ are in $\mathcal{S}'((\mathbb{R}^4)^n)$

(ii) $\mathcal{P}^+_\uparrow$-invariance:

$$\mathcal{W}_n((f_1)_{(\Lambda,a)} \otimes \cdots \otimes (f_n)_{(\Lambda,a)}) = \mathcal{W}_n(f_1 \otimes \cdots \otimes f_n)$$

for all $f_j \in \mathcal{S}(\mathbb{R}^4)$ and all $(\Lambda,a) \in \mathcal{P}^+_\uparrow$.

(iii) Locality:

$$\mathcal{W}_n(f_1 \otimes \cdots \otimes f_j \otimes f_{j+1} \otimes \cdots \otimes f_n) = \mathcal{W}_n(f_1 \otimes \cdots \otimes f_{j+1} \otimes f_j \otimes \cdots \otimes f_n)$$

if $f_j \perp f_{j+1}$.

(iv) Hermiticity:

$$\overline{\mathcal{W}_n(f_1 \otimes \cdots \otimes f_n)} = \mathcal{W}_n(\overline{f_1} \otimes \cdots \otimes \overline{f_n})$$

for all $f_j \in \mathcal{S}(\mathbb{R}^4)$ (order of entries reversed on right hand side).

(v) Spectrum condition:

Use translation invariance to conclude that there are distributions $\mathcal{W}_n$ in the relative position variables $\xi_j = x_j - x_{j-1}$ ($j = 1, \ldots, n-1$) so that

$$\mathcal{W}_n(x_1, \ldots, x_n) = \mathcal{W}_n(\xi_1, \ldots, \xi_{n-1})$$

in formal notation for distributions. Then show that the Fourier transforms $\tilde{\mathcal{W}}_n$ of $\mathcal{W}_n$ have the property that

$$\tilde{\mathcal{W}}_n(q_1, \ldots, q_{n-1}) = 0 \quad \text{if} \quad (q_1, \ldots, q_{n-1}) \notin J^+(0)^{n-1}$$

where $J^+(0)$ denotes the forward directed causal cone emanating from $q = 0$ in Fourier space (often denoted by $V^+$ in the literature).
Problem 4.2
Let $\mathcal{W}_n$ ($n \in \mathbb{N}$) be the Wightman functions of the quantized scalar Klein-Gordon field. Show that the $\mathcal{W}_n$ are determined by the 2-point function $\mathcal{W}_2$ as follows:

(i) If $n$ is odd, then $\mathcal{W}_n = 0$.

(ii) If $n = 2m$ is even, then

\[ \mathcal{W}_n(f_1 \otimes \cdots \otimes f_n) = \sum_{\text{pairings}} \mathcal{W}_2(f_{k_1} \otimes f_{\ell_1}) \cdots \mathcal{W}_2(f_{k_m} \otimes f_{\ell_m}) \]

where the sum is over all ways of writing the set $\{1, 2, \ldots, 2m\}$ as $\{k_1, \ldots, k_m\} \cup \{\ell_1, \ldots, \ell_m\}$ under the condition that

\[ k_1 < k_2 < \ldots < k_m \quad \text{and} \quad k_1 < \ell_1, \ldots, k_m < \ell_m. \]

Hint: You need how the quantized field arises from creation and annihilation operators and their commutation relations. Prove the argument recursively by induction on $n$.

Problem 4.3
For the quantized Klein-Gordon field $\phi$, one can form the unitary groups $e^{it\phi(f)}$, $t \in \mathbb{R}$, (“Weyl operators”) for real-valued test functions $f$.

(i) It holds that

\[ e^{it\phi(f)} e^{is\phi(g)} = e^{it+is\phi(f,g)} e^{i\phi(tf+sg)} \]

for all $s, t \in \mathbb{R}$ and real-valued test functions $f$ and $g$, with a bilinear form $b(\cdot, \cdot)$ on the space of test functions. Determine $b$ in terms of the 2-point function $\mathcal{W}_2$.

(ii) Show that

\[ (\Omega, e^{it\phi(f)} \Omega) = e^{-t^2 \mathcal{W}_2(f,f)/2} \]

holds for all $t \in \mathbb{R}$ and all real-valued test functions $f$ where $\Omega$ is the (Fock-space) vacuum vector of the quantized Klein-Gordon field. Using this, deduce that the $n$-point functions are determined by the 2-point function (cf. Problem 4.2).

Problem 4.4
Let $\mathcal{W}_n$ ($n \in \mathbb{N}$) be the Wightman functions of the quantized scalar Klein-Gordon field. Show that they fulfill spacelike clustering, i.e. for any spacelike vector $a$, it holds that

\[ \lim_{r \to \infty} \mathcal{W}_{t+m}(g_1 \otimes \cdots g_t \otimes \langle f_1 \rangle_{(1,ra)} \otimes \cdots \otimes \langle f_m \rangle_{(1,ra)}) = \mathcal{W}_t(g_1 \otimes \cdots g_t) \mathcal{W}_m(f_1 \otimes \cdots \otimes f_m) \]

for any choice of test functions $g_j, f_k$.  

2