Quantum Field Theory — Problem Sheet 3

Problem 3.1

Define \( \tilde{f}(k) = \int e^{i\eta(k,x)} f(x) \, d^4x \) and define the quantized Klein-Gordon field operators

\[
\phi(f) = \frac{1}{2}(a(\tilde{f}) + a^+(\tilde{f})), \quad f \in \mathcal{S}(\mathbb{R}^4),
\]

where the annihilation/creation operators are defined in \( \mathcal{F}_+ (L^2_{\text{H}^+_m + m}) \), on their natural domain \( \mathcal{F}_{+ \text{ fin}} \) of Fock space vectors having only finitely many \( n \)-particle sectors different from 0.

Show that the following holds.

(i) \( \phi((\Box + m^2)f) = 0, \quad f \in \mathcal{S}(\mathbb{R}^4) \)

(ii) \( \phi(f \circ (\Lambda, a)^{-1}) = U(\Lambda, a)\phi(f)U(\Lambda, a)^{-1}, \quad f \in \mathcal{S}(\mathbb{R}^4), \) for all \( (\Lambda, a) \in \mathcal{P}_+ \), where \( U(\Lambda, a) \) is the 2nd quantization of the unitary representation of the irreducible unitary positive energy representation \( U(\Lambda, a) \) \( ((\Lambda, a) \in \mathcal{P}_+) \) which is on \( L^2_{H^+_m + d\Omega_m} \) given by \( U(\Lambda, a)\tilde{\chi}(k) = e^{i\eta(k,a)}\tilde{\chi}(\Lambda^{-1}k) \).

(iii)

\[
[\phi(f), \phi(g)] = \frac{1}{4} (\tilde{f}, \tilde{g})_{H^+_m} - (\tilde{g}, \tilde{f})_{H^+_m}
\]

for all \( f, g \in \mathcal{S}(\mathbb{R}^4) \).

(iv) The subspace spanned by all vectors of the form

\[
\Omega, \quad \phi(f_1)\Omega, \quad \phi(f_1)\phi(f_2)\Omega, \quad \ldots \quad \phi(f_1)\phi(f_2)\cdots\phi(f_n)\Omega,
\]

where \( n \in \mathbb{N}, f_j \in \mathcal{S}(\mathbb{R}^4), \) and \( \Omega \) is the vacuum vector in Fock space, coincides with \( \mathcal{F}_{+ \text{ fin}} \) and is therefore a dense subspace of \( \mathcal{F}_+ (L^2_{H^+_m + d\Omega_m}) \).

(v) Let \( \Psi, \Psi' \in \mathcal{F}_{+ \text{ fin}} \). Show that

\[
f \mapsto (\Psi, \phi(f)\Psi')_{\mathcal{F}_+}
\]

is a distribution in \( \mathcal{S}(\mathbb{R}^4) \).

(vi) Give an expression for

\[
(\Omega, \phi(f)\phi(g)\Omega)_{\mathcal{F}_+}
\]

in terms of the scalar product of \( L^2_{H^+_m + d\Omega_m} \), where again \( \Omega = (1, 0, 0, \ldots) \) is the vacuum vector in Fock space.