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Quantum Field Theory — Problem Sheet 2

Problems 2.1 to 2.3

Problem 2.1

Let $m > 0$ and $H_m^+ = \{(p_0, \boldsymbol{p}) : p_0 = \sqrt{|\boldsymbol{p}|^2 + m}\}$. For any continuous, compactly supported function $f : H_m^+ \to \mathbb{C}$, define

$$\int_{H_m^+} f(p)\, d\mu(p) = \int_{\mathbb{R}^3} f(p_0, \boldsymbol{p}) \frac{d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2 + m^2}}$$

Show that the measure $d\mu$ defined on H_m^+ defined in this way is invariant under $\mathscr{L}_+^\uparrow$, i.e. that

$$\int_{H_m^+} f(\Lambda p)\, d\mu(p) = \int_{H_m^+} f(p)\, d\mu(p)$$

holds for all $\Lambda \in \mathscr{L}_+^\uparrow$ and all f as specified above.

Hint: Recall any $\Lambda \in \mathscr{L}_+^\uparrow$ can be written as product of a Lorentz boost and a rotation, and prove the statement separately for rotations and for Lorentz boosts.

Problem 2.2

A general theorem in functional analysis asserts that the operator $(-\Delta + m^2)^{1/2}$ on $\mathscr{S}(\mathbb{R}^n)$ is *antilocal*, i.e. for any open subset set G of $\mathbb{R}^n$, the set of all functions

$$f + (-\Delta + m^2)^{1/2} g\,, \quad \text{where } f, g \in C_0^\infty(G)\,,$$

is dense in $L^2(\mathbb{R}^n)$. [I.E. Segal, R.W. Goodman, "Anti-locality of certain Lorentz-invariant operators", J. Math. Mech. **14**, 629-638 (1965)].

Use this result to show that a positive energy solution φ (assumed C^∞) to the Klein-Gordon equation with mass-term $m^2 > 0$ cannot have compactly supported Cauchy data. (The Cauchy-data of a solution φ to the Klein-Gordon equation at $x^0 = \xi$ are the pair of functions $u_0(\boldsymbol{x}) = \varphi(x^0 = \xi, \boldsymbol{x})$, $u_1(\boldsymbol{x}) = \partial_{x^0}\varphi(x^0, \boldsymbol{x})|_{x^0 = \xi}$ defined for $\boldsymbol{x} \in \mathbb{R}^3$.)

Problem 2.3

Let $\mathcal{H}$ be a Hilbert space with bounded hermitean operators A, B and C. Suppose that C commutes with A and B. Show that

$$\mathrm{e}^{i(A+C)t} B \mathrm{e}^{-i(A+C)t} = \mathrm{e}^{iAt} B \mathrm{e}^{-iAt}$$

holds for all $t \in \mathbb{R}$.