Problem 1.1
Show that $SL(2, \mathbb{C})$ is the universal covering group of the proper orthochronous Poincaré group $\mathcal{L}^+_+$, with the covering map $\Lambda(\cdot) : SL(2, \mathbb{C}) \to \mathcal{L}^+_+$ given by

$$\Lambda_{\mu\nu}(A) = \frac{1}{2} \text{Tr}(A\sigma_\mu A^*\sigma_\nu)$$

where $\sigma_0 = 1$ is the $2 \times 2$ unit matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices.

For the proof, proceed along the following steps:

1. Show that there is a one-to-one correspondence between coordinate vectors $x = (x^\mu)_{\mu=0,...,3}$ in Minkowski spacetime and hermitean $2 \times 2$ matrices $H_x$ given by

$$H_x = x^\mu \sigma_\mu, \quad x^\mu = \eta^{\mu\nu} \text{Tr}(H_x \sigma_\nu)$$

where $(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$ and the Einstein summation is employed, i.e. doubly appearing indices (one of them downstairs, the other upstairs) are summed over.

2. Show that

$$\det(H_x) = \eta_{\mu\nu} x^\mu x^\nu, \quad \frac{1}{2} (\det(H_x + H_y) - \det(H_x) - \det(H_y)) = \eta_{\mu\nu} x^\mu y^\nu.$$ (The 2nd equation results from the first by applying the parallelogram identity to symmetric bilinear forms such as the Minowski product $\eta(x, y) = \eta_{\mu\nu} x^\mu y^\nu$.)

3. Use the previous findings to show that for any $A \in SL(2, \mathbb{C})$ there is some proper, orthochronous Lorentz transformation $\Lambda(A)$ such that

$$AH_x A^* = H_{\Lambda(A)x}.$$ (The $2\times2$ matrices are given by the equation above.

4. Show that $\Lambda(A) \Lambda(B) = \Lambda(AB)$, $\Lambda(1_{2 \times 2}) = 1_{4 \times 4}$, $\Lambda(A) = \Lambda(B) \Rightarrow A = \pm B$, and that the matrix $\Lambda(A)$ is given by the equation above.
You may use the fact that $SL(2, \mathbb{C})$ is simply connected to conclude that (i) $\mathcal{L}_+^t$ is not simply connected and (ii) $SL(2, \mathbb{C})$ is the universal covering group of $\mathcal{L}_+^t$. If you like, you can also show that $SL(2, \mathbb{C})$ is simply connected.

**Problem 1.2**

Denote by $\tilde{f}(p) = (2\pi)^{-2} \int e^{-i\eta(p,x)} f(x) d^4x$ the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^4)$, using the Minkowski product in the argument of the phase.

Let $m > 0$ be a fixed number and define for $f, g \in \mathcal{S}(\mathbb{R}^4)$,

$$W(f, g) = C \int_{\mathbb{R}^3} \tilde{f}(-\omega(p), -p) \tilde{g}(\omega(p), p) \frac{d^3p}{\omega(p)}$$

where $C > 0$ is a constant and $\omega(p) = \sqrt{|p|^2 + m^2}$.

(a) Show that $W$ has the properties of a distribution in $\mathcal{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$.

(b) Show that $W(\overline{f}, f) \geq 0$.

(c) Show that $W(f_{(\Lambda,a)}, g_{(\Lambda,a)}) = W(f, g)$ for all $f, g \in \mathcal{S}(\mathbb{R}^4)$, where

$$f_{(\Lambda,a)}(x) = f((\Lambda, a)^{-1}x) \quad (x \in \mathbb{R}^4)$$

for all $(\Lambda, a) \in \mathcal{P}_+^t$.

**Problem 1.3**

For some complex Hilbert space $\mathcal{H}$, $\bigvee^n \mathcal{H}$ and $\bigwedge^n \mathcal{H}$ denote the $n$-fold symmetrized, resp. $n$-fold antisymmetrized tensor product Hilbert spaces of $\mathcal{H}$; by convention, $\bigvee^0 \mathcal{H} = \bigwedge^0 \mathcal{H} = \mathbb{C}$. Then one defines $\mathcal{F}_\pm(\mathcal{H})$, the bosonic (+) / fermionic (−) Fock space on $\mathcal{H}$, as the infinite direct sum Hilbert spaces

$$\mathcal{F}_+(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \bigvee^n \mathcal{H}, \quad \mathcal{F}_-(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \bigwedge^n \mathcal{H},$$

i.e. the spaces consist of sequences $\psi = (\psi_n)_{n=0}^{\infty}$ with

$$\psi_n \in \bigvee^n \mathcal{H} \quad \text{or} \quad \psi_n \in \bigwedge^n \mathcal{H} \quad \text{according to case},$$

and with $(\psi, \psi)_\tau < \infty$, where

$$(\psi, \chi)_\tau = \sum_{n=0}^{\infty} (\psi_n, \chi_n)_n$$

and $(\psi_n, \chi_n)_n$ denotes the scalar product in the appropriate (anti-)symmetrized $n$-fold tensor product Hilbert spaces.

One then defines bosonic creation/annihilation operators $a^+(\chi) / a(\chi)$ in $\mathcal{F}_+(\mathcal{H})$ on the
domain $\mathcal{D}$ of all $\psi = (\phi_n)_{n=0}^{\infty}$ where $\psi_n \neq 0$ only for finitely many $n$ by linear extension of the maps

$$
\frac{1}{\sqrt{n} + 1} a^+(\chi)(f_1 \vee \cdots \vee f_n) = \chi \vee f_1 \vee \cdots \vee f_n,
$$

$$
\sqrt{n} a(\chi)(f_1 \vee \cdots \vee f_n) = (\chi, f_1) f_2 \vee \cdots \vee f_n + \cdots + (\chi, f_n) f_1 \vee \cdots \vee f_{n-1},
$$

$$
a(\chi) f_0 = 0 \quad (f_0 = \psi_0 \in \mathbb{C} = \sqrt{0}{\mathcal{H}})
$$

Similarly, one defines fermionic creation/annihilation operators $b^+(\chi) / b(\chi)$ in $\mathcal{F}_-(\mathcal{H})$ by linear extension of

$$
\frac{1}{\sqrt{n} + 1} b^+(\chi)(f_1 \wedge \cdots \wedge f_n) = \chi \wedge f_1 \wedge \cdots \wedge f_n,
$$

$$
\sqrt{n} b(\chi)(f_1 \wedge \cdots \wedge f_n) = (\chi, f_1) f_2 \wedge \cdots \wedge f_n - (\chi, f_2) f_1 \wedge \cdots \wedge f_n - \cdots - (\chi, f_n) f_1 \wedge \cdots \wedge f_{n-1},
$$

$$
b(\chi) f_0 = 0 \quad (f_0 = \psi_0 \in \mathbb{C} = \bigwedge^0{\mathcal{H}})
$$

In all cases, $\chi$ is in the 1-particle Hilbert space $\mathcal{H}$. The summands on the right hand side of the definition of $b(\chi)$ have alternating signs.

Prove that the following holds.

(a) $(a(\chi) \psi, \psi')_\mathcal{F} = (\psi, a^+(\chi) \psi')_\mathcal{F}$ for all $\psi^{(\ell)} = (\psi_n^{(\ell)})_{n=0}^{\infty} \in \mathcal{D}$,

(b) $[a(\chi), a(\eta)] = 0 = [a^+(\chi), a^+(\eta)], \ [a(\chi), a^+(\eta)] = (\chi, \eta) \cdot 1$ for all $\chi, \eta \in \mathcal{H}$, with the commutator $[X,Y] = XY - YX$

(c) $\{b(\chi), b^+(\eta)\} = 0 = \{b^+(\chi), b^+(\eta)\}, \ \{b(\chi), b^+(\eta)\} = (\chi, \eta) \cdot 1$ for all $\chi, \eta \in \mathcal{H}$, with the anti-commutator $\{X,Y\} = XY + YX$.

(d) $(b(\chi))^* = b^+(\chi)$

(e) $b(\chi)$ and $b^+(\chi)$ are bounded.