On Entropy-Like Invariants for Dynamical Systems

Thomas de Paly

Department of Physics and NTZ of Karl-Marx-University Leipzig, DDR-701 Leipzig, GDR

Abstract. This paper carries over to the theory of the Kolmogorov-Sinai-entropy a method being a basic tool in the theory of the order-structure of states. The function $h(x)=-x\log x$ is replaced by arbitrary bounded, concave functions in all definitions of the entropy-theory. This procedure leads to a class of isomorphy invariants, thus generalizing the notion of dynamical entropy. The general properties of the generalized dynamical entropies are investigated and an explicit calculation of the new invariants is accomplished in some simple cases.

February 1980

KMU-QFT-1 (1980)

L 279/80 III/8/1 192

0. Introduction

In 1958 Kolmogorov introduced the notion of dynamical entropy into the ergodic theory of dynamical systems [3]. He showed with the aid of the entropy that there are nonisomorphic Bernoulli shifts. In 1969 Ornstein solved the isomorphy problem for the class of all Bernoulli shifts by showing that all Bernoulli shifts with the same entropy are isomorphic [8,9]. The isomorphy question for K-systems, however, is unsolved at present [10]. Kouchnirenko constructed generalizations of the K-entropy, the so-called sequence entropies. It was shown by Newton that they give new information about the isomorphy of transformations only in the zero-entropy case [7]. Versik [15] introduced the notion of the scale of a transformation, and Juzvinskij [2] proved that for any positive entropy there are countably many subclasses of K-systems with pairwise distinct scale.

All the isomorphy invariants listed above and a large number other ones are successfully used in the ergodic theory, but all invariants known to the author cannot completely solve the isomorphy question for the class of systems with positive entropy, especially for K-systems. Therefore one needs new invariants.

1. Content of this paper

We construct simple generalizations of the entropy using the following idea. The K-entropy H(T) is defined as a supremum over all finite partitions $\underline{C}$ of the relative entropies

$H(\underline{C}/T)$ of the transformation T with respect to $\underline{C}$. ($H(\underline{C}/T)$ is called in this paper "entropy of the process $(\underline{C}/T)$".) The entropy of a process is well defined, because it is the limit of the entropy of the partitions $\underline{C} \vee T^{-1}\underline{C} \vee \dots \vee T^{-n}\underline{C}$ divided by n, which always exists. The reason for the existence of the limit lies in the subadditivity of the entropy of partitions, i.e. $H(\underline{C} \vee \underline{D}) \leq H(\underline{C}) + H(\underline{D})$. The subadditivity is a consequence of special properties of the function $h(x) = -x\log x$ used in the definition of the entropy [1]. It is well known [1,12] that

$$H(\underline{C}/T) := \lim_n \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\underline{C}) = \lim_n H(\underline{C}/\bigvee_{i=1}^{n} T^{-i}\underline{C}) \quad . \tag{1.1}$$

The existence of the right-hand side of (1.1), however, is guaranteed only by the concavity and boundedness of the function h(x) (and h(0)=0).

Consequently, if we replace h(x) by arbitrary concave, bounded functions g of the closed unit interval with g(0)=0 and then repeat all constructions of the entropy-theory starting with $H(\underline{C}/T) = \lim H(\underline{C}/\bigvee_{i=1}^{n} T^{-i}\underline{C})$, we get a large class of new isomorphy invariants.

In this paper we give the construction and some properties of these new invariants. We compute them explicitly for the cases of zero and infinite entropy ($H(T)=0 \Longrightarrow G(T)=g(1)$ for all g, $H(T)=\infty \Longrightarrow G(T)=\lim_{x \downarrow 0} g'(x)$ for all continuous g if the limit exists).

These extreme cases cannot give new information, but it is shown that for the case of finite, positive entropy

$$g(1) < G(T) < \lim_{x \downarrow 0} g'(x) \tag{1.2}$$

We show for a special class of concave functions that $G(T) < \lim_{x \downarrow 0} g'(x)$ provided the entropy is not too large.
All these statements depend on the entropy, but the results available at present give the hope to get new information for the solution of the isomorphy problem of the dynamical systems from the invariants constructed in this paper.

2. Basic notations and definitions see [1,11,16]

Let $(X,\mathcal{B},\mu)$ be a Lebesgue space, where $\mathcal{B}$ is the σ-algebra of all measurable sets, and μ is a probability measure on $(X,\mathcal{B})$. A mapping $T: X \longmapsto X$, measure-preserving and one-to-one a.e. is called an automorphism of $(X,\mathcal{B},\mu)$.
An aggregate $(X,\mathcal{B},\mu,T)$ with T being an automorphism of the Lebesgue space $(X,\mathcal{B},\mu)$ will be referred to as a dynamical system.

Definition 1

1) Dynamical systems $(X',\mathcal{B}',\mu',T')$ and $(X,\mathcal{B},\mu,T)$ are isomorphic iff there is a mapping $I: X' \longmapsto X$ measure-preserving and one-to-one a.e. (i.e. I is a measure space isomorphism) such that $IT'=TI$ a.e.
2) If I is not a measure space isomorphism but a measure space homomorphism such that $IT'=TI$ a.e., then $(X,\mathcal{B},\mu,T)$ is said to be a factor of $(X',\mathcal{B}',\mu',T')$.
3) A dynamical invariant is a property of dynamical systems which is invariant under isomorphisms of dynamical systems.

The following definitions are concerned with partitions of $(X,\mathcal{B},\mu)$ and sub-σ-algebras of $\mathcal{B}$. All relations between partitions and σ-algebras are understood to hold only up to measure zero. We write $\underline{C} \vee \underline{D}$ for the common refinement of the partitions $\underline{C}$ and $\underline{D}$. For a finite family $\{\underline{C}^i\}_{i=n}^m$ the symbol $\bigvee_{i=n}^m \underline{C}^i$ denotes the partition being the common refinement of all the partitions $\underline{C}^i$. If $n=-\infty$ or $m=\infty$, then $\bigvee_{i=n}^m \underline{C}^i$ is the smallest σ-algebra containing all the listed partitions. If $\underline{C} \vee \underline{D} = \underline{D}$, we write $\underline{C} \leq \underline{D}$.
Analogously, if $\{\mathcal{A}_i\}_{i=n}^m$ is a family of sub-σ-algebras, we denote with $\bigvee_{i=n}^m \mathcal{A}_i$ the smallest sub-σ-algebra of $\mathcal{B}$ containing all $\mathcal{A}_i$.
Let $\mathcal{A}$ be a sub-σ-algbra of $\mathcal{B}$ and $\underline{C}$ a partition. We write $\sigma(\underline{C}) = \mathcal{A}$ iff $\underline{C}$ generates $\mathcal{A}$. The one-to-one correspondence between the set of finite partitions and the set of finite subalgebras contained in the relation $\sigma(\underline{C}) = \mathcal{A}$ is freely used troughout this paper.

Definition 2

Let $g: [0,1] \longrightarrow R$ be a real, bounded, continuous, concave function of the closed unit interval, and let $g(0)=0$. Let further $(X,\mathcal{B},\mu)$ and $\underline{C}$ be a Lebesgue space and a finite partition, respectively. We define

1) $G(\underline{C}) := \sum_i g(\mu(C_i)) \qquad \underline{C}=(C_1,\ldots,C_n)$ (2.1)
2) For any measurable set $A \in \mathcal{B}$, $\underline{C}/A := (C_1 \cap A,\ldots,C_n \cap A)$ is a partition of A induced by $\underline{C}$. The measure $\mu(\cdot)$ induces a probability measure $\mu(\cdot/A)$ on A (A being

a set of positive measure) by

$$\mu(B/A):=\frac{\mu(B\cap A)}{\mu(A)} \qquad \forall B\in\mathcal{B} \tag{2.2}$$

We define

$$G(\underline{C}/A):=\sum_i g(\mu(C_i/A)) \tag{2.3}$$

and

$$G(\underline{C}/\underline{D}):=\sum_j \mu(D_j)G(\underline{C}/D_j) \tag{2.4}$$

where $\underline{D}=(D_1,\ldots,D_m)$ is a second finite partition of X. (See remark 1)

3) Let $\mathcal{A}\subset\mathcal{B}$ be a sub-σ-algebra of $\mathcal{B}$.

$$G(\underline{C}/\mathcal{A}):=\inf_{\underline{D}} G(\underline{C}/\underline{D}) \tag{2.5}$$

$\underline{D}$ runs over all finite partitions with elements in $\mathcal{A}$.

Remarks

1) We use the following convention. If $A\in\mathcal{B}$ is a set of measure zero, then we set $\mu(A)g(\mu(B/A))=0$ $\forall B\in\mathcal{B}$. This is no further restriction on g because $\mu(B\cap A)\le\mu(A)$, and g is bounded on $[0,1]$. With this convention we have a correct definition in equ.2.4 .

2) If we take the function $h(x)=\begin{cases}-x\log x & x\in(0,1]\\ 0 & x=0\end{cases}$

which is continuous, bounded, and concave we get the definitions of the entropy theory. $H(\underline{C})=\sum h(\mu(C_i))$ is the entropy of the partition $\underline{C}$. $H(\underline{C}/\underline{D})$ and $H(\underline{C}/\mathcal{A})$ are the relative entropies of the partition $\underline{C}$ with respect to the partition $\underline{D}$ and the sub-σ-algebra $\mathcal{A}$, respectively.

3. Generalizations of the K-entropy

We want to construct new dynamical invariants along the lines of the entropy theory. To this end we need the following statements on generalized relative entropies.

Proposition 1

Let $\underline{C},\underline{C}^1,\underline{C}^2,\underline{D},\underline{D}^1,\underline{D}^2$ be finite partitions and let $G(\cdot/\cdot)$ be the functionals of definition 2.2. Then

1) $G(\underline{C}/\underline{D})\geq g(1)$ (3.1)

2) $\underline{C}^1\le\underline{C}^2 \Longrightarrow G(\underline{C}^1/\underline{D})\le G(\underline{C}^2/\underline{D})$ (3.2)

3) $\underline{D}^1\le\underline{D}^2 \Longrightarrow G(\underline{C}/\underline{D}^1)\ge G(\underline{C}/\underline{D}^2)$ (3.3)

4) $\underline{C}\le\underline{D} \Longrightarrow G(\underline{C}/\underline{D})=g(1)$ (3.4)

5) If g is strongly concave, $\underline{C}\le\underline{D}\Longleftrightarrow G(\underline{C}/\underline{D})=g(1)$ (3.5)

6) $G(\underline{C}/\underline{D})\le G(\underline{C})$ (3.6)

7) $G(\underline{C}/\underline{D})\le\lim_{x\downarrow 0} g(x)$ (if the limit exists) (3.7)

Proof

We use the results of lemma A.1 (Appendix).

1) $G(\underline{C}/D_j)=\sum_i g(\mu(C_i/D_j))\geq g(\sum_i \mu(C_i/D_j))=g(1)$ (3.8)

2) For each i, $C_i^1=\bigcup_{l_i} C^2_{l_i}$, and therefore

$$g(\mu(C_i^1/D_j))=g(\sum_{l_i}\mu(C^2_{l_i}/D_j))\le\sum_{l_i} g(\mu(C^2_{l_i}/D_j)) \tag{3.9}$$

3) For each j, $D_j^1=\bigcup_{l_j} D^2_{l_j}$, and therefore

$$\mu(D_j^1)g\left(\frac{\mu(C_i\cap D_j^1)}{\mu(D_j^1)}\right)=\left(\sum_{1_j}\mu(D_{1_j}^2)\right)g\left(\sum_{1_j}\mu(C_i\cap D_{1_j}^2)/\sum_{1_j}\mu(D_{1_j}^2)\right)\geq$$

$$\geq\sum_{1_j}\mu(D_{1_j}^2)g\left(\mu(C_i\cap D_{1_j}^2)/\mu(D_{1_j}^2)\right) \qquad (3.10)$$

4) For any j, there is one and only one i such that $D_j\subset C_i$. Therefore $\mu(C_i/D_j)=0$ for all but one $i=\hat{i}$, and $\mu(C_{\hat{i}}/D_j)=1$ This leads to

$$G(\underline{C}/D_j)=g(1) \quad \forall j \qquad (3.11)$$

5) Because of A.1.2, we have $G(\underline{C}/D_j)=g(1)$ if and only if $\mu(C_i/D_j)=0$ for all but one $i=\hat{i}$ and $\mu(C_{\hat{i}}/D_j)=1$. This is equivalent to $\underline{C}\leq\underline{D}$ (up to measure zero).

6,7) are obvious. /=/

Proposition 2

Let $\underline{C}$ be a finite partition and $\{\mathcal{A}_n\}_1^\infty$ be an increasing sequence of sub-σ-algebras of $\mathcal{B}$ (i.e. $\mathcal{A}_n\subseteq\mathcal{A}_{n+1}$ $\forall n$). If $\sigma(\underline{C})\subseteq\bigvee_{n=1}^{\infty}\mathcal{A}_n$ then $G(\underline{C}/\mathcal{A}_n)\xrightarrow{n\to\infty}g(1)$.

Proposition 2 is a modified version of corollary 4.8 of [16]. A sketch of the proof will be given in the Appendix (for details see [11]).

Definition 3

Let $(X,\mathcal{B},\mu,T)$ be a dynamical system. If $\underline{C}$ is a partition of X we call the pair $(\underline{C}/T)$ a process in $(X,\mathcal{B},\mu,T)$ Let $(\underline{C}/T)$ be a process with $\underline{C}$ being a finite partition.

We define
$$G(\underline{C}/T):=\lim_n G\left(\underline{C}/\bigvee_{i=1}^{n}T^{-i}\underline{C}\right) \qquad (3.12)$$
for all functionals defined as in def.2.2.

Proposition 3

The limit in equ.3.12 exists for all processes $(\underline{C}/T)$.

Proof $\bigvee_{i=1}^{n}T^{-i}\underline{C}=T^{-n}\underline{C}\vee\left[\bigvee_{i=1}^{n-1}T^{-i}\underline{C}\right]\geq\bigvee_{i=1}^{n-1}T^{-i}\underline{C}$. Therefore we get $G(\underline{C}/\bigvee_{i=1}^{n}T^{-i}\underline{C})\leq G(\underline{C}/\bigvee_{i=1}^{n-1}T^{-i}\underline{C})$. The sequence of the relative entropies is monotonously decreasing and bounded from below by $g(1)$. /=/

Definition 4

Let $(X,\mathcal{B},\mu,T)$ be a dynamical system. We define for all functionals according to def.3.

$$G(T):=\sup_{\underline{C}} G(\underline{C}/T) \qquad (3.13)$$

$\underline{C}$ runs over all finite partitions measurable $\mathcal{B}$.

Theorem 4

All $G(T)$ in def.4 are dynamical invariants.

Proof The theorem is clear from the definitions because all mappings involved are measure preserving and 1-1 a.e., and sets of measure zero can be neglected because of $g(0)=0$.

Remarks

1) Actually the theorem holds if we use arbitrary real

functions in the definitions 2,3,4. The point however is that for a bounded concave function g the definition of G(T) makes sense. Only in this case we can be sure to have a supremum over well-defined objects (proposition 3).

2) The dynamical entropy (K-entropy) is a special case of def.4. Therefore the G(T) are called generalized dynamical entropies.

In the rest of the paper we derive some properties of the new dynamical invariants. The general properties of the invariants are the content of this section, but in section 4 we deal with a special class of concave functions.

<u>Theorem 5</u>

Let $(X', \mathcal{B}', \mu', T')$ be a factor of $(X, \mathcal{B}, \mu, T)$, then we have for all invariants G

$$G(T') \leq G(T) \qquad . \qquad (3.14)$$

<u>Proof</u> The transformation T' is isomorphic to T restricted to a T-invariant sub-σ-algebra $\mathcal{A}_T \subset \mathcal{B}$. Therefore

$$G(T') = G(T|_{\mathcal{A}_T}) = \sup_{\underline{D} \subset \mathcal{A}_T} G(\underline{D}/T|_{\mathcal{A}_T}) = \sup_{\underline{D} \subset \mathcal{A}_T} G(\underline{D}/T) \leq \sup_{\underline{C} \subset \mathcal{B}} G(\underline{C}/T). \quad /=/$$

<u>Corollary 6</u>

If $(X', \mathcal{B}', \mu', T')$ and $(X, \mathcal{B}, \mu, T)$ are weakly isomorphic (i.e. each system is a factor of the other) then $G(T') = G(T)$ for all invariants G.

A partition $\underline{C}$ is said to be a <u>generator</u> for T iff $\bigvee_{-\infty}^{\infty} T^{-i}\underline{C} = \mathcal{B}$ (up to measure zero). Kolmogorov's theorem which says that for any generator $\underline{C}$ $H(\underline{C}/T) = H(T)$ does not hold in general for the G's. We have instead

<u>Corollary 7</u>

Suppose T has finite generators. Then $G(T) = \sup_{\tilde{\underline{C}}} G(\tilde{\underline{C}}/T)$, where $\tilde{\underline{C}}$ runs over all finite generators of T.

<u>Proof</u> Let $\underline{C}$ be not a generator. Then $\bigvee_{-\infty}^{\infty} T^{-i}\underline{C} = \mathcal{A} \subset \mathcal{B}$ and $T\mathcal{A} = \mathcal{A}$. Therefore $T|_{\mathcal{A}}$ is a factor of T.

$$G(\underline{C}/T) \leq \sup_{\underline{D} \subset \mathcal{A}} G(\underline{D}/T|_{\mathcal{A}}) = G(T|_{\mathcal{A}}) \leq G(T) \qquad /=/$$

<u>Remark</u>

The existence of finite generators is guaranteed for ergodic automorphisms of finite entropy [5].

The aim of the next statements is to compute the new invariants explicitly or to give estimations of them. To do this we need the following lemmas.

<u>Lemma 8</u>

Let $\underline{C}$ be a partition such that $\bigvee_{i=0}^{\infty} T^{-i}\underline{C} = \mathcal{B}$. Then $G(\underline{C}/T) = g(1)$ for all functionals defined in def.3 .

<u>Proof</u> We use proposition 2. Let $\mathcal{A}_n = \sigma(\bigvee_{i=1}^{n} T^{-i}\underline{C})$. $\{\mathcal{A}_n\}$ is an increasing sequence of sub-σ-algebras.

$\bigvee_{i=1}^{\infty} T^{-i}\underline{C} = T^{-1}\bigvee_{i=0}^{\infty} T^{-i}\underline{C} = T^{-1}\mathcal{B} = \mathcal{B}$. Consequently, because of $\sigma(\underline{C}) \subset \mathcal{B}$ we have $\sigma(\underline{C}) \subset \mathcal{B} = \bigvee_{i=1}^{\infty} T^{-i}\underline{C} = \bigvee_{i=1}^{\infty} \mathcal{A}_n$ (up to measure zero).

Therefore prop.2 holds. /=/

Definition 5

We say that a functional $G(\underline{C}/\mathcal{A})$ (cf. def.2.3) has the martingale-property iff for all increasing sequences $\{\mathcal{A}_n\}$ of sub-σ-algebras of $\mathcal{B}$ and for all finite partitions $\underline{C}$

$$\lim_n G(\underline{C}/\mathcal{A}_n) = G(\underline{C}/\bigvee_n \mathcal{A}_n) . \qquad (3.15)$$

Lemma 9

Let g be strongly concave, and assume the functional $G(\underline{C}/\mathcal{A})$ constructed with the function g has the martingale-property. Then

$$G(\underline{C}/T)=g(1) \Longleftrightarrow \sigma(\underline{C}) \subset \bigvee_{i=1}^{\infty} T^{-i}\underline{C} \qquad (3.16)$$

Proof We have $G(\underline{C}/T)=G(\underline{C}/\bigvee_{i=1}^{\infty} T^{-i}\underline{C})$ from the martingale-property. Prop.1.5 leads to $G(\underline{C}/T)=g(1) \Longrightarrow \underline{C} \subset \bigvee_{i=1}^{\infty} T^{-i}\underline{C}$.
The inverse conclusion follows from prop.2 . /=/

Corollary 10

For all dynamical systems $(X,\mathcal{B},\mu,T)$ with entropy $H(T)=0$ and for all generalized dynamical entropies $G(T)=g(1)$ holds.

Proof The relative entropy has the martingale-property (see e.g. [13],Th.4.28). The rest follows from prop.2 . /=/

The suppositions of lemma 8 can be shown to hold for any finite partition in the case of transformations with discrete spektrum. So we find $G(T)=g(1)$ without explicit use of the entropy for these systems.

Corollary 10 says that all the new invariants give the same information in the case of zero entropy, i.e. they are trivial if the entropy is given to be zero.

From the point of view of the entropy the opposites of the dynamical systems with $H(T)=0$ are the Bernoulli shifts. The following proposition gives an estimation for the generalized entropies for Bernoulli shifts. A Bernoulli shift is defined as a shift on $X=\{1,\dots,n\}^{Z}$ ($\mathbb{Z}$ being the set of all integers) with a measure μ given by a product measure on the cylinder sets (see e.g. [16]).

Proposition 11

Let T be the $(q_1 \dots q_n)$-Bernoulli shift with the entropy $H(T)=\sum_{i=1}^{n} h(q_i)=s$. For all invariants G

$$X_g(s) := \sup_{\{p_i\}\in \mathcal{J}} \sum g(p_i) \leq G(T) \leq \lim_{x \downarrow 0} g'(x) \qquad (3.17)$$

holds. $\mathcal{J}$ denotes the set of all probability vectors $\{p_i\}$ with $\sum h(p_i)=s$.

Proof The upper bound is clear from prop.1.7.
The partition $\underline{C}=(A_o^1,\dots,A_o^n)$ (A_o^i is the cylinder set of all elements of X which have the value i on the 0-th place. $\mu(A_o^i)=q_i$ by def.) is a generator for T, and $\underline{C}$ is inde-

pendent of $\underline{D}^n = \bigvee_{i=1}^{n} T^{-i}\underline{C}$ for all n. Therefore

$$G(\underline{C}/T) = \lim_n G(\underline{C}/\underline{D}^n) = \sum_j \mu(D_j^n) G(\underline{C}/D_j^n) = \sum_j \mu(D_j^n) G(\underline{C}) = \sum g(q_i) \; .$$

According to Ornstein's theorem [8] we find for any probability vector $\{p_j\} \in \mathcal{J}$ an independent generator $\tilde{\underline{C}}$ for T consisting of sets $\tilde{C}_j$ with $\mu(\tilde{C}_j) = p_j$. The proposition new follows from the definition of G(T). /=/

Corollary 12

Let T be an ergodic automorphism of positive entropy $H(T) > 0$ Then the inequalities 3.17 hold. Moreover, if g is not identically zero then $X_g(H(T)) > g(1)$.

Proof The first assertion is a simple consequence of Sinai's weak isomorphism theorem [12] and theorem 5. The second is obvious. /=/

We computed an explicit (but trivial) result for the zero entropy case and an estimation (depending on the entropy) for the case of positive entropy. Now we are going to calculate the new invariants for ergodic automorphisms of infinite entropy. This leads to a trivial result again.

Corollary 13

Suppose T is ergodic and let $H(T) = \infty$. Then for all generalized dynamical entropies $G(T) = \lim_{x \downarrow 0} g'(x)$ holds provided the limit exists.

Proof We consider the following sequence of Bernoulli shifts. T_n is the $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$-Bernoulli shift. Then, T_n is a factor of T_m iff $n \leq m$, and all the T_n (n=1,2,...) are factors of T . Therefore we have

$$\sup_{\{p_i\} \in \mathcal{J}_n} \sum g(p_i) \leq G(T_n) \leq G(T) \leq \lim_{x \downarrow 0} g'(x) \quad . \qquad (3.18)$$

Here $\mathcal{J}_n$ denotes the set of all probability vectors with $\sum h(p_i) = \log n$. But $q = (\frac{1}{n}, \ldots, \frac{1}{n}) \in \mathcal{J}_n$ and therefore $\sum g(q_i) = n g(\frac{1}{n}) \leq G(T)$.

Now $\lim_n n\left[g(\frac{1}{n}) - g(0)\right] = g'(0)$ and $g(0) = 0$ complete the proof. /=/

At the end of this section we formulate the obvious

Proposition 14

All generalized dynamical entropies G are monotonously increasing functions of the entropy on the class of all Bernoulli shifts.

Proof We get the desired result if we combine Ornstein's [8,9] and Sinai's [12] isomorphism theorems and theorem 5. /=/

This result is not surprising because the entropy completely determines the isomorphy classes of Bernoulli shifts and is itself contained in the new family of isomorphy invariants. Proposition 14 reflects the fact that no more information on the isomorphy of Bernoulli shifts is to expect if we knew the entropy. The monotonicity results from the fact that all G-invariants are defined in the same manner as a supremum over partitions.

4. A special class of invariants

The new dynamical invariants constructed in the previous section are very hard to compute in nontrivial cases because there is no analogue of Kolmogorov's theorem. To find further nontrivial properties we consider a special family of concave functions having a simple structure. We define

$$g_r(x) = \begin{cases} x & 0 \leq x \leq r \\ r & x > r \end{cases} \qquad 0 < r < 1 \qquad (4.1)$$

and denote all functionals associated with g_r with G_r. In a forthcoming paper [11] we will use the G_r to construct an example showing that in 3.17 equality doesn't hold in general even for the case of Bernoulli shifts. Here we only want to answer the question whether there are dynamical systems with $G_r(T) < 1 = g'(0)$ or not, thus proving the existence of systems with $G_r(T)$ between the trivial values $g(1) = r$ and $g'(0) = 1$.

Proposition 15

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system with entropy $0 < H(T) = s < \infty$ and assume $r < e^{-s}$. Then $G_r(T) < 1$.

Remark

The logarithm in the function $h(x) = -x\log x$ is taken to the basis e. If another basis b is used then proposition 15 holds if $r < b^{-s}$.

Proof of prop.15 $\quad G_r(T) = \sup_{\underline{C}} G_r(\underline{C}/T)$. We can restrict ourselves to processes $(\underline{C}/T)$ with $H(\underline{C}/T) = s$. For $r < e^{-s}$ there exist reals $\varepsilon, \delta, \gamma > 0$ such that $r = e^{-(s+\varepsilon)} - \gamma$ and $(s+\varepsilon)(1-\delta) > s$. We denote $\underline{D}^n = \bigvee_{i=1}^{n} T^{-i}\underline{C}$. Because of $H(\underline{C}/\underline{D}^n) \downarrow s$ there is a $n_0(\underline{C}, \varepsilon, \delta)$ such that $\forall n > n_0$

$$s \leq H(\underline{C}/\underline{D}^n) < (s+\varepsilon)(1-\delta) \qquad (4.2)$$

Assume $n > n_0$. We show that in the partition $\underline{C} \vee \underline{D}^n$ there are some elements $C_i \cap D_j^n$ the union of which has a measure greater than δ and which have the property

$$\mu(C_i \cap D_j^n) > e^{-(s+\varepsilon)} \mu(D_j^n) \qquad (4.3)$$

The elements of $\underline{C} \vee \underline{D}^n$ not fulfilling (4.3) are denoted by $C_k \cap D_l^n$.

Assume $\sum_{i,j} \mu(C_i \cap D_j^n) \leq \delta$. Then we have for the sets $C_k \cap D_l^n$ because of $\mu(C_k \cap D_l^n) \leq e^{-(s+\varepsilon)} \mu(D_l^n)$

$-\log \mu(C_k/D_l^n) \geq s+\varepsilon$. This in turn leads to

$-\sum_{\substack{k \neq i \\ l \neq j}} \mu(C_k \cap D_l^n) \log \mu(C_k/D_l^n) \geq (s+\varepsilon)(1-\delta)$. By the positivity of $h(x)$ and the definition of the relative entropy we get

$$H(\underline{C}/\underline{D}^n) \geq (s+\varepsilon)(1-\delta)$$

This contradicts (4.2). Therefore $\sum_{i,j} \mu(C_i \cap D_j^n) > \delta$.

From $r < e^{-(s+\varepsilon)}$ we get $g_r(\mu(C_i/D_j^n)) = r$ and $g_r(\mu(C_k/D_l^n)) \leq \mu(C_k/D_l^n)$.

Now $r = e^{-(s+\varepsilon)} - \gamma$ leads to

$$G_r(\underline{C}/\underline{D}^n) \leq \sum_{k,l} \mu(C_k \cap D_l) + \sum_{i,j} \mu(D_j^n)\left[\mu(C_i/D_j^n) - \gamma\right] =$$

$$= 1 - \gamma \sum_{i,j} \mu(D_j^n) < 1 - \gamma\delta \, .$$

Therefore $G_r(\underline{C}/T) < 1 - \gamma\delta < 1$ for all finite partitions $\underline{C}$, and the upper bound is independent of $\underline{C}$. So it holds for the supremum as well. /=/

6. Discussion

We constructed a large class of dynamical invariants by generalizing the notion of K-entropy. The construction is based on the idea to replace the function $h(x)=-x\log x$ by an arbitrary concave, bounded function with $g(0)=0$. This method stems from the theory of the order-structure of states [14] which has been used successfully in the analysis of the irreversible behaviour of physical systems.

The dynamical entropy can be computed for many systems. The theorem of Kolmogorov being a basic tool for the computation of the entropy does not hold in general for the new invariants. Therefore the explicit calculation of the G(T) seems to be a very hard problem, and results are known only in some cases which are trivial from the point of view of the entropy theory. One sees however that any invariant constructed with a function g not being identically zero is nontrivial, i.e. one can find dynamical systems $(X,\mathcal{B},\mu,T),(X',\mathcal{B}',\mu',T')$ such that $G(T) \neq G(T')$.

Whether there are systems with equal entropy and $G(T) \neq G(T')$ for some g is unknown at present, although it seems that

this should be true. This conjecture is sustained by the non-validity of an analogue of Kolmogorov's theorem on the entropy of a generating process. The new invariants are trivial for systems of zero entropy, but in this case there are invariants such as sequence entropy and support [6] the properties of which are not yet completely investigated and which are trivial for K-systems. So the dynamical invariants presented inthis paper could become a useful supplement to the entropy theory provided the difficulties of the explicit calculation can be superseded.

At the moment there is no hope to find general methods for the computation of all new invariants. Therefore as a first step the properties of the class of invariants constructed with the functions g_r are considered. The G_r take values which are not trivial (i.e. there are systems T with $g(1)=r<G_r(T)<1=g'(0)$). Moreover, from the study of an example we know that even in the Bernoullian case the supremum of $G_r(\underline{C}/T)$ over the independent generators can be smaller than $G_r(T)$ [11]. This example while illustrating the non-triviallity of the new invariants, brings out the deep difficulties connected with the explicit computation of the generalized entropies.

We already noticed that the construction of the G-invariants is based on an idea from the theory of the order-structure of states. This raises the question whether there is a structure in the set of all automorphisms of a measure space which is induced by the generalized K-entropies. Theorem 5 gives a first hint, but more interesting is a study of the consequences of $G(T) \leq G(T')$ for all g. It is to expect that this leads to a physical interpretation of the new invariants which is still an open problem.

Appendix 1. Concave functions on $[0,1]$

A function $g: [0,1] \longrightarrow R$ is called concave iff $\forall x,y,\lambda \in [0,1]$

$$g(\lambda x+(1-\lambda)y) \geq \lambda g(x)+(1-\lambda)g(y)$$

If the equality holds only for $x=y$ and (or) ($\lambda=0$ or $\lambda=1$), g is called a strongly concave function.
The following properties of concave functions are easily verified [11,14].

Lemma A.1

Let $g: [0,1] \longrightarrow R$ be concave and continuous, and let $g(0)=0$

1) $g(x)+g(y) \geq g(x+y) \quad \forall x,y \in [0,1]$ with $x+y \leq 1$

2) If g is strongly concave, $g(x)+g(y)=g(x+y)$ holds if and only if $x=0$ and (or) $y=0$.

3) $g(x) \leq x \cdot \lim_{y \downarrow 0} g'(y)$ if the limit exists. g' denotes the first derivative of the function g.

4) Let $(s_i),(t_i)$ be sequences of the same length, $s_i,t_i \geq 0$ $\forall i$, $\sum s_i < \infty$. Then

$$\left(\sum_i s_i\right) g\left(\sum t_i / \sum s_i\right) \geq \sum_i s_i g(t_i/s_i)$$

Appendix 2. Proposition 2 (Sketch of the proof)

Proposition 2 is a generalization of corollary 4.8 of [16]. This corollary holds for the relative entropy of a partition with respect to a sub-σ-algebra of $\mathcal{B}$ and is widened to the functionals of def.2.3. The proof of the generalized version follows exactly the line of the cited statement in [16].

Lemma A.2

Let $(X,\mathcal{B},\mu)$ and $\mathcal{B}_0 \subset \mathcal{B}$ be a probability space and an algebra, resp. Assume that $\mathcal{B}_0$ generates the σ-algebra $\mathcal{B}$. If $\underline{C}$ is a finite partition measurable $\mathcal{B}$, then for all $\varepsilon > 0$ and for any continuous, bounded, concave function $g: [0,1] \longrightarrow R$ with $g(0)=0$ there is a finite subalgebra $\mathcal{D} \subset \mathcal{B}_0$ such that

i) $g(1) \leq G(\underline{C}/\mathcal{D}) < g(1)+\varepsilon$

ii) $g(1) \leq G(\underline{D}/\mathcal{A}) < g(1)+\varepsilon$

where $\mathcal{A} = \sigma(\underline{C})$ and $\mathcal{D} = \sigma(\underline{D})$.

For the proof we have only to remark that the lower bounds are trivial and that there is a $0 < \delta_0 < 1$ such that

i) $0 \leq x < \delta_0 \Longrightarrow -\frac{\varepsilon}{r} < g(x) < \frac{\varepsilon}{r}$

ii) $1-\delta_0 < x \leq 1 \Longrightarrow g(1)-\frac{\varepsilon}{r} < g(x) < g(1)+\frac{\varepsilon}{r}$.

r is the number of sets the partition $\underline{C}$ consists of.
This simple fact is the only additional argument to the proof of th.4.8 in [16] which is the entropy version of the lemma. Lemma A.2 leads directly to prop.2 by using the same arguments as in the cited corollary 4.8.

Acknowledgements

I wish to express my thanks to Prof.Dr.Uhlmann for the encouragement during the completion of this work. I am also indebted to Dr.Alberti and Dr.Crell for helpful discussions.

References

[1] Arnold,V.I.; Avez,A. Problèmes ergodiques de la mécanique classique Gauthier-Villars Paris 1967

[2] Juzvinskij,S.A. Различение K-автоморфизмов шкалой Func.an. i pril. 7/4 1973 70...75

[3] Kolmogorov,A.N. Новый метрический инвариант транзитивных динамических систем и автоморфизмов Dokl.Ak.Nauk SSSR 119/5 1958 861...864

[4] Kouchnirenko,A.G. О метрических инвариантах типа энтропии Usp.math.nauk 11/5 1967 57...65

[5] Krieger,W. On entropy and generators of measure-preserving transformations Trans.Am.Math.Soc. 149/2 1970 453...464

[6] Michel,H. Support as an invariant for dynamical systems Preprint MLU Halle 1979

[7] Newton,D. On sequence entropy I,II Math.Syst.Th. 4 1970 119...128

[8] Ornstein,D.S. Bernoulli shifts with the same entropy are isomorphic Adv.Math. 4 1970 337...352

-.- , Two Bernoulli shifts with infinite entropy are isomorphic Adv.Math. 5 1970 339...348

[9] Ornstein,D.S. Эргодическая теория, случайность и динамические системы (Transl. from Engl.) Isdatelstvo Mir Moscow 1978

[10] Ornstein,D.S.; Shields,P. An uncountable family of K-automorphisms Adv.Math. 10/1 1973 63...88

[11] de Paly,T. On a class of generalized K-entropies Preprint KMU QFT Leipzig 1980

-.- , Eine neue Klasse K-Entropie-ähnlicher Invarianten in der Ergodentheorie Thesis Leipzig 1980

[12] Sinai,I.G. Слабый изоморфизм преобразований с инвариантной мерой Dokl.Ak.Nauk SSSR 147/4 1962 797...800

[13] Smorodinsky,M. Ergodic theory, entropy Lect.Notes Math. 214 Springer-Verlag 1971

[14] Alberti,P.; Uhlmann,A. Stochasticity and partial order to appear Verlag der Wissenschaften Berlin

nn,A. Ordnungsstrukturen im Raum der gemischten nde und die H-Theoreme von Felderhoff-van Kampen wiss.Z.KMU Leipzig M.N.R. 27/3 1978 213...228

[15] Versik,A.M. Метрический инвариант автоморфизмов пространства с мерой Dokl.Ak.Nauk SSSR 209/1 1973 15...18

[16] Walters,P. Ergodic theory - Introductory lectures Lect.Notes Math. 458 Springer-Verlag 1975