STOCHASTIC LINEAR MAPS AND TRANSITION PROBABILITY

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ABSTRACT. Some aspects of the transition probability $P(\omega, \nu)$ between states ω, ν on unital *-algebras are discussed. It is shown that P increases under the action of any stochastic linear map T, i.e., $P(T\omega, T\nu) \geqslant P(\omega, \nu)$. Some properties of P are derived in starting from a recently-proved characterization of the quantity in question.

1. INTRODUCTION

There is a very natural extension of the quantum theoretical notation of 'transition probability' to states given by density matrices or even more general mixed states. Computationally, if ψ_1 and ψ_2 denote normalized Schrödinger functions of a (say large, many-body) quantum system, and if ρ_1, ρ_2 denote the matrices of the corresponding reduced states of a (perhaps small, few-body) sub-system, then

$$|(\psi_1, \psi_2)| \leqslant \mathrm{Tr}\,(\rho_1^{1/2}\rho_2\rho_1^{1/2})^{1/2}. \tag{1.1}$$

Thus, we get a reasonable estimation from above of $|(\psi_1, \psi_2)|$. The square of the right-hand side of (1.1) is called the 'transition probability of the two states given by the density matrices ρ_1, ρ_2'. As has been proved by Uhlmann in [1], the assertions above are valid and consistent with the following general definition:

DEFINITION. Let $\mathscr{A}$ be a unital *-algebra and u_1, u_2 two of its states. By $P(u_1, u_2)$ we denote the smallest number satisfying the inequality

$$|(\psi_1, \psi_2)|^2 \leqslant P(u_1, u_2) \tag{1.2}$$

for every unital *-representation π of $\mathscr{A}$, which represents u_1 and u_2 as vector states:

$$\forall a \in \mathscr{A} : u_j(a) = (\psi_j, \pi(a)\psi_j). \tag{1.3}$$

(The unitality of π can be replaced by the requirement that ψ_1, ψ_2 are normalized vectors.)

We get an immediate corollary of this definition in considering a unital *-homomorphism Φ

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from one unital *-algebra $\mathscr{B}$ into another one, $\mathscr{A}$. Using Φ, every representation of $\mathscr{A}$ can be lifted to one of $\mathscr{B}$. The associated dual map Φ^* transforms states of $\mathscr{A}$ into states of $\mathscr{B}$ and we always have

$$P(u_1, u_2) \leqslant P(\Phi^* u_1, \Phi^* u_2). \tag{1.4}$$

In the following we shall deal only with unital C^*-algebras and W^*-algebras since more general *-algebras have not yet been studied systematically in that context.

2. GENERAL PROPERTIES OF THE TRANSITION PROBABILITY

Let us start with a theorem, the proof of which is given in [7]:

THEOREM 1. *Let $\mathscr{A}$ be a unital C^*-algebra and u_1, u_2 two of its states. Then*

$$P(u_1, u_2) = \inf_a u_1(a) u_2(a^{-1}) \tag{2.1}$$

where the infinum runs through all positive and invertible elements a of $\mathscr{A}$.

That it is possible to represent the transition probability, as given by (2.1), is an important enough property of P by itself. Besides, the fact stated by Theorem 1 opens exceedingly comprehensive ways for deriving all the other properties of P, as will be demonstrated in the following.

REMARKS. (1) The concavity properties of P proved in [1], see also [2], can be directly read off from (2.1).

(2) Let us assume the existence of a state u and of elements, b_1, $b_2 \in \mathscr{A}$, with $u_j(a) = u(b_j^* a b_j)$, $j = 1, 2$, for all a, such that $b_1^* b_2 = b_2^* b_1 \geqslant 0$. Then $P(u_1, u_2) = u(b_1^* b_2)^2$, according to Uhlmann in [1] and the proof implicitly contains Theorem 1 in this special case. More recently, Alberti in [7] has shown how to systematically compute P by representation-theoretic formulae. Also, Theorem 1 has been proved there in its most general form; the proof being heavily based upon a result of Araki and Raggio in [3].

(3) That the right-hand side of (2.1) is larger or equal to its left-hand side is true for every *-algebra and follows at once from the definition of P. The nontrivial part is in the opposite inequality.

As a first application of Theorem 1, we are now going to derive a new transformation property for P.

Let us consider two unital C^*-algebras, $\mathscr{A}$ and $\mathscr{B}$. A linear map $T: \mathscr{A}^* \mapsto \mathscr{B}^*$ is called *stochastic* iff it transforms every state of $\mathscr{A}$ into a state of $\mathscr{B}$. If a map $S: \mathscr{B} \mapsto \mathscr{A}$ happens to exist such that for all states u of $\mathscr{A}$ and all elements b of $\mathscr{B}$ we have $(Tu)(b) = u(Sb)$, then S is uniquely determined by T. We then say T_* exists and write $S = T_*$. If T is stochastic, then T_* is a unital, positive linear map and vice versa.

THEOREM 2. *Let $\mathscr{A}$, $\mathscr{B}$ be unital C^*-algebras, T: $\mathscr{A}^* \mapsto \mathscr{B}^*$ a linear stochastic map, and u_1, u_2 states of $\mathscr{A}$. Then*

$$P(u_1, u_2) \leqslant P(Tu_1, Tu_2). \tag{2.2}$$

Proof. Let us first assume that existence of T_*. Then, according to (2.1), we have with the positive and invertible elements $b \in \mathscr{B}$

$$P(Tu_1, Tu_2) = \inf u_1(T_*b)u_2(T_*b^{-1}). \tag{2.3}$$

As following from Kadison's inequality (cf. [5]), Choi in [6] has obtained

$$T_*b^{-1} \geqslant (T_*b)^{-1} \tag{2.4}$$

for every unital positive T_*. Hence, $P(Tu_1, Tu_2) \geqslant \inf u_1(T_*b)u_2((T_*b)^{-1})$, and the infimum is extended over all positive invertible $a \in \mathscr{A}$ having a representation $a = T_*b$. Now, by Theorem 1, we arrive at (2.2). If T_* does not exist we use a variant of the 'bidual trick' as following: T induces a unital positive map T^* from $\mathscr{B}^{**}$ to $\mathscr{A}^{**}$. Then we consider the stochastic map S with $S_* = T^*$, apply the reasoning above and trace back to the property (2.2) of T with the aid of the following simple fact (see [4] or [7]): if u_1, u_2 are states of a unital C^*-algebra $\mathscr{A}$ and v_1, v_2 are their unique normal extensions to the bidual $\mathscr{A}^{**}$, then $P(u_1, u_2) = P(v_1, v_2)$. □

In [4], another proof has been given, not depending on Theorem 1, which assumes T_* to be *completely* positive. In using Kadison's inequality and Theorem 3 below, this result has been refined by Alberti [7] by showing 2-positivity of T_* to be sufficient.

Surprisingly, we now find that even 2-positivity is not necessary for the validity of (2.2)!★ Another nice application of Theorem 1 gives the following

THEOREM 3. *Given two states u_1, u_2, of a unital C^*-algebra $\mathscr{A}$. We denote by Q the set of all linear functionals $v \in \mathscr{A}^*$ satisfying*

$$\forall a,\ b \in \mathscr{A}: |v(a^*b)|^2 \leqslant u_1(a^*a)u_2(b^*b). \tag{2.5}$$

Then

$$P(u_1, u_2) = \sup_Q |v(1)|^2 \tag{2.6}$$

For the *proof* we return to the very definition of P. From (1.2) and (1.3) we infer: $a \mapsto (\psi_1, \pi(a)\psi_2)$ is contained in Q. This comes from Schwarz's inequality, and tells us the correctness of '$\leqslant$' in (2.6). On the other hand, given $v \in Q$ and a positive and invertible, we replace in (2.5) a by $a^{1/2}$ and b by $a^{1/2}$. This gives $|v(1)|^2 \leqslant u_1(a)u_2(a^{-1})$. Theorem 1 tells us the correctness of ' ' (2.6), and we have arrived at a reasonably concise derivation of the result in question (Cf. this added to the

★The somewhat restrictive remark of [4] in this respect is simply due to a mistake.

proof given in [7], also provides, however, some other interesting features of transition probability.)

As another example we shall demonstrate how useful Theorem 1 can be when applied for calculating transition probabilities in infinite quantum systems. To this aim, let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \ldots$ an increasing sequence of unital *-subalgebras (not necessarily norm-closed ones) such that their union is norm-dense in $\mathscr{A}$. Given two states, u, v, of $\mathscr{A}$, we denote by u_j, v_j their restrictions onto $\mathscr{A}_j$. Since $\mathscr{A}$ = norm-closure $\cup_j \mathscr{A}_j$, and in the *Hermitian part* of $\mathscr{A}$ the invertible, positive elements constitute an open set, the union of all invertible and positive elements of all $\mathscr{A}_j$ has to be norm-dense within the former set. Hence, Theorem 1 implies

$$P(u, v) = \inf u(a)v(a^{-1}) = \inf_j \inf_{a_j} u(a_j)v(a_j^{-1}), \tag{2.7}$$

where a_j runs through all positive, invertible elements of $\mathscr{A}_j$. Applying Theorem 1 to (2.7) we see the following result:

THEOREM 4.

$$P(u, v) = \inf_j P(u_j, v_j). \tag{2.8}$$

Once more, we again remark that there is another way of proving (2.8) which does not make reference to Theorem 1 but starts from (2.6).

3. ONCE MORE ON A TRANSFORMATION PROBLEM

In this section let us come back to our starting point, which we mentioned in the introduction. Let us look on a 'large' quantum system, characterized by the Hilbert space H. We assume only some subsystem of our large system is accessible, or of interest, to our observations, this 'small' subsystem being attached to a separable Hilbert space H_1. By H_2 we mean the Hilbert-space which represents the 'external world' or the surrounding of the small system within the large one. In the following we want to assume that the 'bath' system is not a finite quantum system, i.e., H_2 is supposed to be dimensionally infinite, and $H = H_1 \otimes H_2$. Let ψ_1, ψ_2 be two normalized vectors of H describing possible quantum-mechanical states of the whole system. Then, the states of our small system are (without further information) only accessible to a quantum-statistical description, given by the correspondingly reduced density operators ρ_1, ρ_2 over H_1:

$$\rho_1 = \mathrm{Tr}_2\, p_{\psi_1}, \qquad \rho_2 = \mathrm{Tr}_2\, p_{\psi_2} \tag{3.1}$$

where p_{ψ_j} is the one-dimensional orthoprojection over H which corresponds to ψ_j, $j = 1, 2$, and 'Tr_1', 'Tr_2' means the operation of taking the partial trace with respect to H_1, H_2, respectively. Since these operations appear to be restrictions to normal states of certain stochastic maps, we arrive at (1.1) once more if we refer to Theorem 2. Let us now discuss the following problem: given density operators ρ_1, ρ_2 over H_1 and state vectors ψ_1, $\psi_2 \in H$, what relations exist amongst

$\rho_1, \rho_2, \psi_1, \psi_2$ provided the inequality (1.1), i.e., $P(\rho_1, \rho_2) \geqslant |(\psi_1, \psi_2)|^2$, is satisfied? The answer is in the following:

THEOREM 5. *Inequality* (1.1) *holds true if and only if there is a unitary operator u on H such that ρ_1 and ρ_2 appear as reduced density operators of $u\psi_1$ and $u\psi_2$, respectively.*

Proof. Due to unitary invariance of transition probabilities the 'if' part of the assertion follows from our discussions above. To see the other way around, assume $P(\rho_1, \rho_2) \geqslant |(\psi_1, \psi_2)|^2$, and let $\eta \in H_2$, $\varphi_1, \varphi_2 \in H_1$ be normalized vectors with $(\varphi_1, \varphi_2) = (\psi_1, \psi_2)$. There is some unitary v over H such that $\varphi_j \otimes \eta = v\psi_j$, $j = 1, 2$. Let us by p_1, p_2 denote the one-dimensional orthoprojections over H_1 referring to φ_1, φ_2, and assume e is the one-dimensional orthoprojection over H_2 which refers to η. Then, from our assumptions we see $P(\rho_1, \rho_2) \geqslant P(p_1, p_2)$. By a result proved in [4] (Satz 3 there), the preceding inequality suffices to guarantee that there is some unital, normal, completely positive linear map T, over the bounded operators $B(H_1)$ on H_1, such that

$$\forall a \in B(H_1): \operatorname{Tr} \rho_j a = \operatorname{Tr} p_j Ta, \quad j = 1, 2. \tag{3.2}$$

By our assumptions on H, H_1, H_2, some elementary facts on normal *-representations of $B(H_1)$ together with the Stinespring's theorem* imply that

$$\forall a \in B(H_1): Ta = m^*(a \otimes 1_2)m, \tag{3.3}$$

with m being an isometry acting from H_1 into H, with $m^*m = 1_1$.

Let there be defined another isometry $w: H_1 \ni \xi \mapsto \xi \otimes \eta \in H$. Then

$$\forall a \in B(H_1): \operatorname{Tr} \rho_j a = \operatorname{Tr} (p_j \otimes e) wm^*(a \otimes 1_2) mw^*, \tag{3.4}$$

where (3.2) and (3.3) have been used. Defining $wm^* = k \in B(H)$, we may rewrite (3.4) into the form

$$\forall a \in B(H_1): \operatorname{Tr} \rho_j a = \operatorname{Tr} k^*(p_j \otimes e)k(a \otimes 1_2). \tag{3.5}$$

Now, $kk^* = wm^*mw^* = ww^* = 1_1 \otimes e$, hence k is a partial isometry on H and, moreover, $s = (p_1 \vee p_2) \otimes ek$ is a partial isometry over H with a finite-dimensional final projection. This makes especially sure that a unitary operator k' exists over H such that $s = (p_1 \vee p_2) \otimes ek'$. Since $p_j \otimes e = vp_{\psi_j}v^*$, we get with the unitary operator $u = k'^*v$ that

$$k^*(p_j \otimes e)k = s^*(p_j \otimes e)s = k'^*(p_j \otimes e)k' = up_{\psi_j}u^* = p_{u\psi_j},$$

for $j = 1, 2$. Therefore, (3.5) reads as

$$\forall a \in B(H_1): \operatorname{Tr} \rho_j a = \operatorname{Tr} p_{u\psi_j}(a \otimes 1_2) = \operatorname{Tr} (\operatorname{Tr}_2 p_{u\psi_j})a,$$

*For more detailed information and references, the reader is recommended to [8], where these problems are dealt with in the context of quantum theory.

which means $\rho_j = \mathrm{Tr}_2\, p_{u\psi_j}$, $j = 1, 2$. □

Finally, let us state another consequence from all that we have proved:

THEOREM 6. *Suppose ω_1, ω_2 are normal, pure states over the bounded operators $B(H)$ on some Hilbert space H. Assume T is a stochastic linear map which transforms ω_1, ω_2 into normal states $T\omega_1$, $T\omega_2$ over $B(H)$. Then there exists a normal, unital, completely positive linear map T' from $B(H)$ into $B(H)$, such that $T\omega_j = \omega_j \circ T'$, $j = 1, 2$.*

Proof. By Theorem 2 we know $P(T\omega_1, T\omega_2) \geqslant P(\omega_1, \omega_2)$. Since ω_1, ω_2 and $T\omega_1$, $T\omega_2$ correspond to density operators, where the former couple consists of extremal, normal elements, the result of [4] which we already referred to in passing in the proof of Theorem 5, provides T' as asserted. □

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