ON STOCHASTIC MAPS OF C*-ALGEBRAIC STATE SPACES

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1. Some notations, definition of stochastic maps.

We consider C*-algebras, A, B, ..., with an identity which we denote
by 1 or, more carefully, by 1_A, 1_B, ... . Let A* denote the Banach
space of bounded linear functionals of the algebra A. We need the fol-
lowing subsets of A*:

- The set of Hermitian functionals, A_H,
- The cone of positive linear functionals, A_+,
- The convex set of states, Ω_A.

In particular we have u ∈ Ω_A iff u ∈ A_+ and u(1_A) = 1.

Given two C*-algebras, A and B, the Banach space of the bounded
linear maps T : A* → B*

is denoted by Lin(A*, B*). In this space there is a natural weak topo-
logy, the neighbourhoods of which around the identity map are given by

\{ T : |(Tu_1)(b_1)| + ... + |(Tu_m)(b_m)| < ε \}

where u_1, ..., u_m are from A*, and b_1, ..., b_m are elements of B.

One knows that a weakly closed and bounded in the Banach norm of Lin(...) set
is a weakly compact one.

A map T : A* → B* is called positive (or positive preserving) iff

T : A_+ → B_+

The linear map T : A* → B* is called stochastic iff it maps Ω_A
into Ω_B,

T : Ω_A → Ω_B

Stochasticity of T is equivalent with linearity, positivity preserving,
and the normalizing condition (Tu)(1_B) = u(1_A). We call

St(A*, B*)

the set of all stochastic maps from A* into B*.

This set is a weakly compact convex subset of Lin(A*, B*).

We shall distinguish a subclass of stochastic maps. By

St®(A*, B*)

we denote the weak closure of the set of all maps T allowing for a
representation

\[ u \in A^* : T u = \sum_j u(a_j) v_j \]

finite sum,

where v_1, v_2, ... are states of the algebra B, i.e. out of Ω_B,
and a_1, a_2, ... ∈ A such that a_1 + a_2 + ... = 1_A, and

\[ a_j \geq 0 \]

for all j.

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2. An inequality.

At first we consider real-valued functions, \( f = f(s_1, \ldots, s_n) \), defined on \( \mathbb{R}^n \) and satisfying

(a) \( \forall s \geq 0 : f(s s_1, \ldots, s s_n) = s f(s_1, \ldots, s_n) \)

(b) \( f(s_1 + t_1, \ldots, s_n + t_n) \leq f(s_1, \ldots, s_n) + f(t_1, \ldots, t_n) \)

For the purpose of abbreviation we refer to such functions as to \( h \)-convex functions. (Under the condition (a) the subadditivity (b) and convexity are equivalent.)

Let now \( A \) be a C*-algebra with identity element. Choosing on \( \mathbb{R}^n \) an \( h \)-convex function \( f \) we define for every \( n \)-tuple of Hermitian functionals \( u_1, \ldots, u_n \in A^*_n \) the expression

\[
S_f(u_1, \ldots, u_n) := \sup \sum_j f(u_1(a_j), \ldots, u_n(a_j))
\]

where the supremum is running through all the partitions of the identity \( \{ a_1, \ldots, a_m \} \) of \( A \), i.e. through all possible choices of finitely many elements \( a_j \in A \) with \( a_j \geq 0 \) and \( a_1 + a_2 + \ldots + a_m = 1_A \).

The subadditivity of \( f \) is guaranteeing the finiteness of \( S_f \).

**Theorem 1:** Let \( u_1, \ldots, u_n \) denote \( n \) Hermitian functionals from \( A^*_n \).

Then we have

\[
S_f(Tu_1, \ldots, Tu_n) \leq S_f(u_1, \ldots, u_n)
\]

for every \( T \in \text{St}(A^*, B^*) \), and for every \( h \)-convex \( f \).

To prove this we need

**Lemma 1:** Let \( \tilde{u}_1, \ldots, \tilde{u}_n \) be the uniquely defined normal extensions of \( u_1, \ldots, u_n \) to the bidual \( A^{**} \) of \( A \). Then

\[
S_f(\tilde{u}_1, \ldots, \tilde{u}_n) = S_f(u_1, \ldots, u_n)
\]

This statement (see p.53 of [1]) is a simple consequence of the fact that every finite partition of the identity of \( A^{**} \) can be approximated by those of \( A \) in the strong operator topology of \( A^{**} \), see lemma 2.1.2. of [17] where this lemma is reduced to the KAPLANSKI density theorem (compare [27]). Now using the lemma above we proceed as follows.

\( T: A^* \rightarrow B^* \) induces \( T^*: B^* \rightarrow A^{**} \) and \( T^*: (A^{**})^* \rightarrow (B^*)^* \). On the imbeddings

\[
A^* \hookrightarrow (A^{**})^* \quad \text{and} \quad B^* \hookrightarrow (B^*)^*
\]

given by the normal extensions, \( T^{**} \) coincides with \( T \) and because of lemma 1 it suffices to show the inequality

\[
S_f(T^{**}\tilde{u}_1, \ldots, T^{**}\tilde{u}_n) \leq S_f(\tilde{u}_1, \ldots, \tilde{u}_n)
\]

Let us choose \( \delta > 0 \) and a partition of the identity \( \{ b_1, \ldots, b_m \} \) of \( B^{**} \) such that

\[
\delta + S_f(T^{**}\tilde{u}_1, \ldots, T^{**}\tilde{u}_n) \leq \sum f(T^{**}\tilde{u}_1(b_j), \ldots, T^{**}\tilde{u}_n(b_j))
\]
The right hand side equals
\[ \sum f( \tilde{u}_1(T^* b_j), \ldots, \tilde{u}_n(T^* b_j) ) \]  
However, \( T^* \) is a positivity preserving map from \( B^{**} \) into \( A^{**} \) and it is \( T \mathbf{1}_{B^{**}} = \mathbf{1}_{A^{**}} \). Therefore, the expression (x) is smaller than \( S_f(\tilde{u}_1, \ldots, \tilde{u}_n) \).

We may use the proof to obtain a further statement. Let \( T \) be positive and contracting from \( A^* \) into \( B^* \). Then \( T^* \) is positive and contracting from \( B^* \) into \( A^{**} \). Hence \( 1 - \mathbf{1} + T^* b_j = a \) is a positive element. Hence after adding \( f(\tilde{u}_1(a), \ldots, \tilde{u}_n(a)) \) to (x) in this new situation the result will be smaller than \( S_f(\tilde{u}_1, \ldots, \tilde{u}_n) \). The additional term is non-negative provided \( \forall j : u_j \in A^* \), and \( f \geq 0 \) in case of \( \forall j : s_j \geq 0 \). Denoting by \( R^+_n \) the subset of \( R^n \) given by \( \forall j : s_j \geq 0 \) one gets

**Theorem 2:** Let \( T : A^* \rightarrow B^* \) be a positive contraction. Let \( f \) be \( h \)-convex and assume \( f \geq 0 \) on \( R^+_n \).

Then for every \( n \)-tuple of positive linear functionals \( u_1, \ldots, u_n \in A^* \) it is
\[ S_f(\tilde{u}_1, \ldots, \tilde{u}_n) \leq S_f(u_1, \ldots, u_n) \]

**Remark:** Let \( f \) be \( h \)-convex. Then \( f \geq 0 \) on \( R^+_n \) is equivalent with
\[ f(s_1', \ldots, s_n') \geq f(s_1, \ldots, s_n) \]  
if \( \forall j : s_j' \geq s_j \).

Let us hence call
\[ h^\tau \text{-convex} \]  
resp.
\[ h^\ast \text{-convex} \]
every \( h \)-convex function \( f \) on \( R^n \) satisfying
\[ f \geq 0 \]  
resp.
\[ f \leq 0 \]  
on \( R^+_n \)

1. **Existence of stochastic maps.**

We start by constructing certain functionals of type \( S_f \).

Let \( B \) be a \( C^* \)-algebra and \( b_1, \ldots, b_n \) Hermitian elements of it. Given real numbers \( s_1, \ldots, s_n \), the norm of the positive part of \( s_1 b_1 + \ldots + s_n b_n \),
\[ f := \| (s_1 b_1 + \ldots + s_n b_n) \| = \sup_{v \in \Delta_B} v(s_1 b_1 + \ldots + s_n b_n) \]  
(4)
defines, considered as a function on \( R^n \), an \( h \)-convex function.

The more, in case \( \forall j : b_j \geq 0 \), it follows \( h \)-convexity for \( f \), and in case \( \forall j : b_j \leq 0 \) it will follow \( h \)-convexity of \( f \).

Let us now consider a further \( C^* \)-algebra, \( A \), and \( u_1, \ldots, u_n \) some of its Hermitian linear functionals. We write
\[ K(u_1, \ldots, u_n \mid b_1, \ldots, b_n) := S_f(u_1, \ldots, u_n) \]  
with \( f = \| (s_1 b_1 + \ldots + s_n b_n) \| \)
(5)

These particular \( S_f \)-functionals we need in the following.
We rewrite the question, whether there is an inverse to theorem 1 statement in terms of the algebras

\[ A(n) := A \oplus A \oplus \cdots \oplus A \quad \text{and} \quad B(n) := B \oplus \cdots \oplus B, \]

with \( n \) direct summands. Of course, \( A(n) \) is a C*-algebra again, and

\[ A(n)^* = A^* \oplus \cdots \oplus A^*, \]

\( n \) direct summands,

in a natural way. For technical reasons only let us denote by

\[ I_+ \quad \text{and} \quad I_- \]

the following sets of maps

\[ T(n) : A(n)^* \rightarrow B(n)^* \]

which are constructed this way: There is \( T \in \text{St}^0(A^*, B^*) \), and there are \( w_1, \ldots, w_n \in B_n^* \) such that for all \( u_1, \ldots, u_n \in A^* \) we have

\[ T(n) : u_1 \oplus \cdots \oplus u_n \mapsto (Tu_1 + w_1) \oplus \cdots \oplus (Tu_n + w_n). \]

Using this setting we demand

\[ T(n) \in I_+ \text{ iff } \forall j : w_j > 0 \]

\[ T(n) \in I_- \text{ iff } \forall j : -w_j > 0. \]

The concept of weak topology remains relevant in an obvious manner for transformations from \( A(n)^* \) into \( B(n)^* \) not being linear necessarily. \( I_+ \) and \( I_- \) are composed of a weakly compact and a weakly closed set of transformations, and they are both weakly closed, hence.

Therefore the sets

\[ F_+ := I_+ (u_1 \oplus \cdots \oplus u_n) \quad \text{and} \quad F_- := I_- (u_1 \oplus \cdots \oplus u_n) \]

of \( B(n)^* \) are weakly closed convex ones.

Let us at first consider \( F_+ \), and let us assume \( x_1 \oplus \cdots \oplus x_n \notin F_+ \).

Then there exists a weakly continuous Hermitian linear form \( q \) on \( B(n)^* \) separating this element from \( F_+ \), i.e.

\[ q(x_1 \oplus \cdots \oplus x_n) > \sup_{r \in I_+} q(r). \]

Being Hermitian and weakly closed there are Hermitian elements \( b_1, \ldots, b_n \) of \( B_n \) such that

\[ \forall x_1 \oplus \cdots \oplus x_n \in B(n)^* : q(x_1 \oplus \cdots \oplus x_n) = x_1(b_1) + \cdots + x_n(b_n). \]

With this choice of \( b_1, \ldots, b_n \) we get

\[ x_1(b_1) + \cdots + x_n(b_n) > \sup_{r \in I_+} (Tu_1(b_1) + \cdots + Tu_n(b_n)) + w_1(b_1) + \cdots + w_n(b_n). \]

By definition, \( w_1, \ldots, w_n \) are varying freely within \( B_n^* \). Therefore the supremum can and will be finite for \( b_1 \leq 0, \ldots, b_n \leq 0 \), only.

But then the supremum will be reached with \( w_1 = w_2 = \cdots = 0 \) only.

Thus we get

\[ x_1(b_1) + \cdots + x_n(b_n) > \sup_{T \in \text{St}^0(A^*, B^*)} (Tu_1(b_1) + \cdots + Tu_n(b_n)). \]
But $\text{St}^0(A^*,B^*)$ is the weak closure of certain known in their structure maps, and in performing the desired supremum it suffices to take only these. Hence the supremum has to run through all decompositions, $a_1,\ldots,a_m$, of $1_A$ with $m = 1,2,\ldots$, and all choices of $m$ states of $B$ in

$$x_1(b_1)+\ldots+x_n(b_n) > \sup \sum \sum u_j(a_i) v_j(b_j).$$

Performing the supremum with respect of the states $v_1,\ldots,v_m \in \Omega_B$ we get

$$x_1(b_1)+\ldots+x_n(b_n) > \sup \sum \sum (u_j(a_i) b_j)_+.$$  

The right hand side of this inequality equals $K(u_1,\ldots,u_n;b_1,\ldots,b_n)$. We see: $x_1 \oplus \ldots \oplus x_n \in F_+$ if there are $b_1 \in 0, \ldots, b_n \in 0$, fulfilling this inequality. But the reverse, $v_1 \oplus \ldots \oplus v_n \in F_+$, may be expressed by

$$\forall j: v_j = T u_j + w_j, \quad w_j \geq 0, \quad T \in \text{St}^0(A^*,B^*).$$

Thus we have established (we pass from $b_j \leq 0$ to $-b_j, b_j \geq 0$):

**Theorem 2:** Let $A$, $B$ be two $C^*$-algebras with identity elements, and $u_1,\ldots,u_n \in A_h^*$ and $v_1,\ldots,v_n \in B_h^*$.

There is a map $T \in \text{St}^0(A^*,B^*)$ satisfying

$$\forall j = 1,2,\ldots,n: \quad T u_j \leq v_j$$

if and only if for all elements $b_1 \geq 0,\ldots, b_n \geq 0$ out of $B$

$$- [v_1(b_1) + \ldots + v_n(b_n)] \leq K(u_1,\ldots,u_n;b_1,\ldots,-b_n)$$

is valid.

Literally the same reasoning with $F_-$ instead of $F_+$ gives

**Theorem 4:** Let $u_1,\ldots,u_n \in A_h^*$ and $v_1,\ldots,v_n \in B_h^*$ be given where $A$ and $B$ are $C^*$-algebras with identity elements.

There is a map $T \in \text{St}^0(A^*,B^*)$ satisfying

$$\forall j: T u_j \geq v_j$$

iff for all $b_1 \geq 0,\ldots, b_n \geq 0$ out of $B$ we have

$$v_1(b_1) + \ldots + v_n(b_n) \leq K(u_1,\ldots,u_n;b_1,\ldots,b_n).$$

**Corollary:** Assuming

$$\forall j: u_j \in \Omega_A, \quad v_j \in \Omega_B,$$

the existence of a stochastic map out of $\text{St}^0(A^*,B^*)$ with

$$\forall j: T u_j = v_j$$

is equivalent with

$$v_1(b_1) + \ldots + v_n(b_n) \leq K(u_1,\ldots,u_n;b_1,\ldots,b_n)$$

where $b_1,\ldots,b_n \in B$ runs either through all positive elements, or through all negative elements, or through all Hermitian elements of $B$.

This, for commutative $A$ and $B$, has been proved in the same manner by Alberti and Ulhmann [1] as an intermediate result in order to prove the
inverse to theorem 1 statement. It turns out that only the commutativity of $B$ is needed for this purpose, and, indeed, an even slightly weaker assumption. It is our next aim to show this.

**Definition:** Let $B$ be a C*-algebra with identity element.

We shall say the n-tuple $\{v_1, \ldots, v_n\}$, $\forall j: v_j \in B_n^*$, fulfills condition (C), iff for every choice of $n$ Hermitian elements, $b_1, \ldots, b_n$, out of $B$ one has

$$v_1(b_1) + \ldots + v_n(b_n) \leq K(v_1, \ldots, v_n; b_1, \ldots, b_n).$$

(6)

We now see the following: The validity of equ. (2) and (6) implies because of (5) the validity of the assumptions used in theorems 3 and 4. Together with theorem 1 the following is nothing but a mere rewriting of the results already obtained:

**Theorem 5:** Let $A$ and $B$ be two C*-algebras with identity element.

Let $u_1, \ldots, u_n \in A_n^*$ and $v_1, \ldots, v_n \in B_n^*$ be two n-tuples of Hermitian linear forms, and assume the validity of condition (C) for the n-tuple $\{v_1, \ldots, v_n\}$.

Then there exists a stochastic map $T \in St^0(A^*, B^*)$ satisfying

(a) $\forall j: Tu_j \geq v_j$

if and only if for all $h^+$-convex functions, $f$, it is

$$S_f(u_1, \ldots, u_n) \geq S_f(v_1, \ldots, v_n).$$

(b) $\forall j: Tu_j \leq v_j$

if and only if for all $h^-$-convex functions, $f$, it is

$$S_f(u_1, \ldots, u_n) \leq S_f(v_1, \ldots, v_n).$$

We shall now supplement these statements by a more handy than condition (C) assumption.

**Lemma 2:** Let $B$ be a commutative C*-algebra with identity. Then every n-tuple of Hermitian linear forms $v_1, \ldots, v_n \in B_n^*$ fulfills the condition (C).

This lemma is an important particular case of the following

**Conjecture:** Let $B$ be a C*-algebra with identity element, and $\{v_1, \ldots, v_n\}$ a n-tuple out of $B_n^*$.

Let $B_0$ be the following C*-subalgebra of $B$: $b \in B$ is an element of $B_0$ if and only if for every $c \in B$ we have

$$v_j(c b) = v_j(b c), \quad j=1,2,\ldots,n. $$

(7)

Then $\{v_1, \ldots, v_n\}$ satisfies condition (C) if $B_0$ contains a maximal commutative subalgebra of $B$.

From theorems 3 and 4 one concludes the validity of condition (C) if
and only if
\[ \forall j = 1, \ldots, n : \quad T v_j = v_j \quad \text{with suitable} \quad T \in \text{St}^0(B^*, B^*) \] (8)
can be fulfilled. For commutative $B$ this has been proved (see theorem 2.3.3 of [11]) by showing

if $B$ is commutative then: $\text{St}(B^*, B^*) = \text{St}^0(B^*, B^*)$. (9)

In this particular case the identity map is contained in $\text{St}^0$ by (9), and we are allowed to take in (8) the identity map.

Lemma 3: Let $B$ be a $W^*$-algebra, $D \subseteq B_0$ a discrete maximal abelian subalgebra, and $v_1, \ldots, v_n$ normal functionals satisfying (7).

Then the conjecture is true, i.e., \{v_1, \ldots, v_n\} fulfils condition (C).

'D is discrete' is understood as following: There is a family \{p_k\}, indexed by an index set, of projections satisfying

(i) $\sum p_k = \text{identity map}$, where the convergence is the weak one over the directed set of finite subsets of the index set.

(ii) $\forall b \in B, \forall k : p_k b p_k = \delta_k(b) p_k$

with complex numbers $\delta_k(p)$.

Let now $N$ be a finite subset of the index set. Defining the projection $q_N$ by $1 = q_N + \sum p_k$, summation over $\mathbb{N}$, we get a resolution of the identity. Let us choose a state $u$ such that $u(q_N) \neq 0$ and $\forall k \in N : u(p_k) \neq 0$. Then we define $T_N$ by

\[ (T_N v)(b) = u(q_N)^{-1} u(q_N b q_N) v(q_N) + \sum u(p_k)^{-1} u(p_k b p_k) v(p_k) \] (10)

where the summing is over $N$. Obviously, $T_N \in \text{St}(B^*, B^*)$.

Let $v$ be normal. Given $\varepsilon > 0$ there is a finite subset $N$ of the index set such that the norms of $v - \sum_N v(p_k)$ and $v(q_N \cdot q_N)$ are smaller than $\varepsilon$. Because of (ii) it is

$u(p_k)^{-1} u(p_k b p_k) v(p_k) = \delta_k(b) v(p_k) = v(p_k b p_k)$.

If, therefore, $v(b c) = v(c b)$ for all $c \in D$, the sum of the right-hand side of (10) equals $v(b) - v(b q_N)$. Hence $|T_N v(b) - v(b)| < 2 \varepsilon \|b\|$. From this we infer the existence of a weak limit $T$ of the operators $T_N$ with $T v = v$ for all normal states satisfying the assumption of the conjecture. Thus lemma 3 is proved. /\$

Let us now assume $B$ to be a $C^*$-algebra with unit element, and $D$ a maximal commutative $*$-subalgebra. Let us define

$Q = \{ v \in B_0 : v(b d) = v(d b) \quad \text{for all} \quad b \in B, d \in D \}$. (11)

We denote by $W_D$ the restriction of a linear functional $v \in B^*$ onto $D$, and $\|W_D\|$ its functional norm, i.e. its norm as an element of the Banach space $D^*$. 
Theorem 6: Under the condition that
\[ v \in Q : \| v \| = \| v_D \| \] (12)
every finite subset of Q satisfies condition (C).

Proof: Step 1: Let \( w \in D_h^* \). There is a \( v \in Q \) with \( v_D = w \). We need to prove this only for states. Let \( w \) be a state of \( D \) and \( R(w) \) the set of all those states \( v \) of \( D \) the restriction on \( D \) of which is \( w \), i.e. \( v_D = w \). \( R(w) \) is not empty by well known theorems [2]. Further, \( R(w) \) is a weakly compact convex set. The map \( v(\cdot) \rightarrow v(e^{-1} \cdot e) \), where \( e \) denotes a unitary element of \( D \), is a continuous automorphism of \( R(w) \). If \( e \) runs through the unitaries of \( D \) these maps are commuting one with another. Therefore, by the KAKUTANI fixed point theorem, there is \( v_0 \in R(w) \) with \( v_0 = v_0(e^{-1} \cdot e) \) for all unitaries \( e \in D \). From this it follows \( v_0 \in Q \).

Step 2: If \( v, \tilde{v} \in Q \) and \( v_D = \tilde{v}_D \), then \( v = \tilde{v} \). Proof: condition (12) implies
\[ \| v - \tilde{v} \| = \| v_D - \tilde{v}_D \| = 0 \]for \( v_D = \tilde{v}_D \).

Step 3: Let
\[ T_B : (T_B w)(d) = \sum w_j(d) w(d_j) \]
with states \( w_j \) of \( D \) and \( d_1, d_2, \ldots \) a decomposition of the unity in \( D \). Then we have \( T_B \in \text{St}^0(D^*, D^*) \).

As already mentioned there is a net \( \{ T_B \} \) with \( \lim T_B = \text{id.} \) of \( D_h^* \) in the weak topology. Let \( v_j \) be the unique (!) extension of \( w_j \) to \( B \) which is in \( Q \), \( v_j \in Q \), \( (v_j)_B = w_j \). Then
\[ \hat{T}_B = \sum v_j(\cdot) w(d_j) \]
extends \( T_B \) to \( \text{St}^0(B^*, B^*) \). Let \( \hat{v} \) be a weak limit of the net \( \{ \hat{T}_B \} \).

Then for all \( v \in Q \) we have \( (\hat{v})_B = v_D \) and \( \hat{v} \in Q \). But the unique extension of \( v_D \) is \( v \). This implies \( \hat{v} = v \) for all \( v \in Q \).

This, together with \( \hat{v} \in \text{St}^0(B^*, B^*) \), implies the validity of condition (C) for all finite subsets of \( Q \). \(/=/

Remark: If \( B \) is commutative then \( \text{St}(A^*, B^*) = \text{St}^0(A^*, B^*) \), and

every such map is a completely positive one (in the sense of Umegaki and Stinespring). The same, i.e. the complete positivity, is true
for the map constructed in proving lemma 3.

Remark: For pairs of Hermitian functionals \( (n = 2) \) and with \( A = B \)
the n-dimensional commutative \(*\)-algebra of all complex-valued
functions on \( n \) points our theorems are equivalent with various
classical results concerning the existence of stochastic and doubly
stochastic matrices. They are well described in [3 - 8]. A more
recent contribution is in [9] and, besides [1] and the references
cited there, in [10].
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References