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Peter M. Alberti and Armin Uhlmann
Department of Physics, Karl-Marx-University, Leipzig

Stochasticity and Partial Order

Doubly Stochastic Maps and Unitary Mixing

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Introduction

It is our intention to explain and to prove certain relations between stochasticity and partial order within this book.

An elementary but typical example is the following theorem due to RADO. Let us consider two n-tuples of real numbers, $\boldsymbol{a} = \{a_1, a_2, \ldots\}$ and $\boldsymbol{b} = \{b_1, b_2, \ldots\}$, which are decreasingly ordered, i.e. $a_1 \geqq a_2 \geqq \cdots$ and $b_1 \geqq b_2 \geqq \cdots$ respectively, and which fulfil $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Then $\boldsymbol{a}$ is a convex sum of n-tuples $\boldsymbol{b}^{(k)}$, where every $\boldsymbol{b}^{(k)}$ denotes a permutation of the n-tuple $\boldsymbol{b}$, if and only if

$$\forall m: \sum_{j=1}^{m} a_j \leqq \sum_{j=1}^{m} b_j . \tag{1}$$

Using Birkhoff's result concerning the structure of doubly stochastic matrices, one at once infers the equivalence of (1) under the assumptions above with the existence of a doubly stochastic matrix T mapping $\boldsymbol{b}$ onto $\boldsymbol{a}$, i.e. $T\boldsymbol{b} = \boldsymbol{a}$.

For reasons of physical interpretation we call $\boldsymbol{a}$ "more chaotic" than $\boldsymbol{b}$ iff $\boldsymbol{a} = T\boldsymbol{b}$ with a doubly stochastic T. In the first chapter, there are several examples explaining this point. In particular we refer to the increase of all "entropy-like quantities" if probability vectors become more chaotic.

After mainly dealing with now-classical results and some of their recent applications in Chapter 1, we extend these considerations to matrices in Chapter 2. Though there is little known of the structure of doubly stochastic transformations in matrix spaces (see Definition 2-2), one can establish similar connections between such doubly stochastic and related maps and appropriate partial (pre-)orderings. In the first two chapters we select, hopefully, a representative part of our present knowledge in that domain, although several branches cannot even be touched.

In Chapter 3 we indicate how to relate all that to the geometry of state spaces of finite-dimensional $*$-algebras and intend to convince the reader of the inescapability of extending it to general W^*-algebras.

The main tools and proofs for doing so are given in Chapters 4 and 5. We believe part of our techniques to be of general interest in the theory of W^*-algebras; for example, the Ky Fan functionals, the c-ideal, the central-valued convex trace, and the Σ-property.

In the last chapter, we introduce the dual structure to the partial (pre-)ordering $\succ$ of the state space. Let a, b be two Hermitian elements of a W^*-algebra. We write $a \succ b$ iff for all positive linear functionals ω of the algebra, we have

$$\sup \omega(a^u) \geqq \sup \omega(b^u) . \tag{2}$$

Here a^v, b^u denote unitary transforms of the elements a, b, and the supremum has to run through all unitaries of the algebra in question. In this way one gets an extension of the famous von Neumann-Murray classification of pairs of projectors to pairs of Hermitian elements.

We do not present analogues in W^*-algebras for all the theorems of the first two chapters. However, using the technicalities developed, the skilled reader can find many of them by himself. Furthermore, one may obviously extend this theory completely to C^*-algebras with "sufficiently many projections", i.e. to AW^*-algebras. For general C^*-algebras, however, the results are essentially incomplete.

As already mentioned, there are parts of our work that are related to physics, including the description of some irreversible processes, the classification of mixed (in the sense of GIBBS and VON NEUMANN) states, and the problem of general diffusions. There are connections to non-commutative probability theory and non-commutative ergodicity.

Last not least, we point to the following statement: With the exception of the first chapter, the theory presented relies on the use of the unitary transformations of a W^*-algebra. Therefore, without some care, one gets trivialities for commutative algebras. In 1.9. a relevant idea for the commutative case is explained. Its implications for general commutative C^*-algebras will be considered elsewhere.*)

*) P. M. ALBERTI and A. UHLMANN, Dissipative motion in state spaces, Teubner-Texte zur Mathematik Bd. 33, Leipzig 1981.