A note on stochastic dynamics in the state space of a commutative C* algebra

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In this paper a functional characterization of stochastic evolutions within the state spaces of commutative C* algebras with identity is derived. Consequences concerning the structure of those linear evolution equations (master equations) that give occasion to stochastic evolutions are discussed. In part, these results generalize facts which are well known from the finite-dimensional classical case. Examples are given and some important particularities of the W* case are developed.

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1. BASIC NOTIONS AND TOOLS

Let A denote a commutative C* algebra with unity 1. Whereas the C* norm of x ∈ A will be marked by ∥x∥, the functional norm of an element ω ∈ Ω(A) of the topological dual A* of A will be denoted by ∥ω∥. As usually, the state space of A, S(A), is defined as the convex set S(A) = {ω ∈ Ω(A): ω(1) = 1}. Let B(A*) denote the linear space of all bounded linear maps acting from the Banach space A* into A*. Then ∥ω∥, on A* induced norm on elements of B(A*) will be denoted by the same symbol ∥ω∥. Being equipped with this topology, (A*)∗ becomes a Banach algebra. An element Φ ∈ B(A*) is said to be stochastic if Φω(1) = ω(1) and Φ is positive, i.e., Φω ∈ A+ whenever ω ∈ A+. Where A+ means the positive cone in A*. The convex set of all stochastic maps with respect to A will be denoted by ST(A). Let {ϕα}α∈X be a net of elements of B(A*). Then, we say that the net is weakly converging towards ϕ ∈ B(A*), ϕα → ϕ, if limα ϕα(ω) = ϕ(ω) for every ω ∈ A* and each element x of A. It is an important fact that ST(A) is weakly compact. Stochastic maps are exactly those linear transformations on A* that throw states into states. This property makes them very useful for the abstract description of dynamical evolutions of systems (in our case classical systems, for only commutative C* algebras will be under consideration throughout this paper).

In many applications we will meet commutative W* algebras. There, by standard knowledge, we may identify the commutative W* algebra A with L∞(G, μ) for a suitable measure space G with measure μ. In this context, besides the whole set of states, there is the set of normal states deserving our interest. These states belong to the predual Aσ of A. In the sense of the canonical identification from above, Aσ can be identified with L1(G, μ). Thus, normal states correspond to probability distributions over certain measure spaces, and this is the frame in which problems of classical statistical mechanics usually will be dealt with. In this situation, the set of linear transformations that take normal states into normal ones will be referred to as STω(A).

Let f denote a real-valued function on the positive cone R+ ≡ (s1, ..., sn) → f(s1, ..., sn).

We will refer to f as an h-convex function (of order n) if f is finite, continuous, convex, and homogeneous of first degree on R+. We remark that homogeneity and convexity imply that h-convex functions are subadditive on R+. By means of h-convex function f we define a functional Sf on n-tuples of positive linear forms of A by

Sf(ω1, ..., ωn) = sup{a} ∑ f(ω1(a1), ..., ωn(ak)) (1.1)

where the sup runs through the set of decompositions {ak} of 1 into finitely many positive elements of A (i.e., ∑ ak = 1). Sf in this situation will be spoken of as an h-convex functional (of order n), and Sf is called positive if f is non-negative on R+.

In order to get a better idea of an h-convex functional Sf, we will take notice of

Proposition 1.1: Let ω1(x), ..., ωn(x) ∈ L1(G, μ) correspond to positive normal functionals ω1, ..., ωn of the W* algebra L∞(G, μ) Then, every positive h-convex functional Sf can be represented by

Sf(ω1, ..., ωn) = ∫ f(ω1(x), ..., ωn(x))dμ(x) (1.2)

A proof is given in the Appendix to this paper. The importance of h-convex functionals (of arbitrary order) is due to the following result (see Ref. 1):

Theorem 1.2: Let ω1, ..., ωn, σ1, ..., σn be states of the commutative C* algebra A with identity. Then, there exists a stochastic map ϕ ∈ ST(A) performing the transformation

ωk = ϕσσ for k = 1, ..., n (1.3)

if and only if

Sf(ω1, ..., ωn) ≤ Sf(σ1, ..., σn) (1.4)

for every h-convex functional Sf of order n over Aσ. Moreover, the occurrence of (1.4) for all positive h-convex functionals Sf is sufficient to guarantee the existence of ϕ obeying (1.3).

We close our preparations by introducing a relation to between indexed sets of states.

Let N = (ωi)i∈I, N' = (ω'i)i∈I' be two indexed sets (labeled by the same index set) of states on A. Then, we
define

\[ \text{Definition 1.3:} (\forall n) N \ni n', \text{ if, for any natural } n \text{ and every choice } i_1, \ldots, i_n \in \mathbb{N}, \text{ we have} \]
\[ S_f(\omega_{i_1}, \ldots, \omega_{i_n}) \leq S_f(\omega'_{i_1}, \ldots, \omega'_{i_n}) \]
for every \( h \)-convex functional \( S_f \) of order \( n \).

2. THE MAIN RESULT

We start our considerations in fixing the sense of what is called stochastic dynamics. Assume there is given a set of stochastic maps \( (T_{ts}) \), the members \( T_{ts} \) labeled by the pairs \((t, s)\) of non-negative reals \( t, s \) with \( t \geq s \).

\[ \text{Definition 2.1: (stochastic dynamics) } (T_{ts}) \text{ is called stochastic dynamic if} \]
\[ T_{tt} = \text{id} \quad \forall t \geq 0, \quad (2.1) \]
\[ T_{ts} = T_{tt} T_{ts} \quad \forall s \leq t \geq 0, \quad (2.2) \]
\[ T_{ts}(A^*) \text{ is dense in } A^*. \quad (2.3) \]

In case of a \( W^* \)-algebra \( A \), \( (T_{ts}) \) is said to be a normal stochastic dynamic if \( T_{ts} \in \mathcal{ST}(A) \), and (2.3) is replaced with "\( T_{ts}(A) \) is dense in \( A^* \)." A simple example of a stochastic dynamic is given by

\[ \text{Example 2.2: Let } A = \mathbb{L}^1(\{1, \ldots, N\}), \text{ and assume } M = (M_{ik}) \text{ an } N \times N \text{ matrix with properties:} \]
\[ i) \sum_k M_{ik} = 0, \quad \forall k, \]
\[ ii) M_{ii} = 0, \quad \forall i, \quad M_{ik} = 0, \quad \forall i \neq k. \]

Then, \( \{T_{ts} = \exp(M(t-s))\} \) is a stochastic dynamic within \( A^* = \mathbb{L}^1(\{1, \ldots, N\}) \).

Let \( \omega \in S_A \), and define a "trajectory" \( (\omega_t)_{t \geq 0} \) within \( S_A \) by
\[ \omega_t = T_{ts}\omega, \quad t \geq 0. \]

Then, \( \omega = \omega_t \) will be called initial state of the trajectory \( (\omega_t)_{t \geq 0} \) under this stochastic dynamic \( (T_{ts}) \). The total system \( \{\omega_t\}_{t \geq 0} \) of \( \omega \in S_A \) of the trajectories generated by \( (T_{ts}) \) has the following properties
\[ A^* = \{ [\omega_t]_{0 \leq t \leq s} \} \text{ is dense in } A^* \text{ for } t > 0, \quad (2.5) \]
\[ (\omega_t)_{0 \leq t \leq s} \Rightarrow (\omega_s)_{0 \leq s} \text{ whenever } t \geq s \quad (2.6) \]

where in (2.6) we referred to Theorem 1.2 and Definition 1.3 with \( 1 = S_A \), and \([\ ]\) in (2.5) means the operation of taking the linear hull. In the next step, let us ignore the presence of the generating dynamic \( (T_{ts}) \), and extract from (2.5) and (2.6) the following notion:

\[ \text{Definition 2.3: (c-system) Let } S_0 \text{ be a subset of states } A. \text{ We call } \{[\omega_t]_{0 \leq t \leq s} \} \text{ the c-system (of trajectories)} \]
\[ \text{in one case that} \]
\[ A^* = \{[\omega_t]_{0 \leq t \leq s} \} \text{ is dense in } A^* \text{ for all } t > 0, \quad (2.7) \]
\[ \text{where } \omega_0 = \omega \text{ is supposed; } \]
\[ (\omega_t)_{0 \leq t \leq s} \Rightarrow (\omega_s)_{0 \leq s} \quad \forall t \geq s \geq 0. \]
$\Lambda > \Lambda'$ for $\Lambda, \Lambda' \in F(1)$ in case that $\Lambda > \Lambda'$, we may think of $\{F(1), >\}$ as a directed set. Then, for $\Lambda \in F(1)$, we are assured of the existence of $T_s \in ST(\Lambda)$ such that $\omega_s = T_s \omega_t$, $\forall t \in \Lambda$, where we made use of Theorem 1.2. Since $ST(\Lambda)$ is weakly compact, we find a converging subnet $(T_{s,e})_{s,e}$ of the net $(T_s)_{s \in \Lambda}$. Denote the limit by $T$. Assume $\Lambda = A$, and $i > 1$. Then, we find $\beta \in K$ such that $\Lambda \beta > \{i\}$ whenever $i \beta > \beta_0$, thus $(T_{s,\beta})_{\lambda}$

\begin{equation}
(\lambda \lambda \cdot \omega_{s,\beta}(x) = \lim_{\lambda \rightarrow \lambda}(T_{s,\beta}(\omega_{s,\beta})(x) = \lim_{\lambda \rightarrow \lambda}(T_{s,\beta}(\omega_s)(x) = \omega_s(x) \text{ by definition of } T_{s,\beta}.
\end{equation}

This latter happens for every $x \in A$, so $T_{s,\beta} \omega_s = \omega_s$ has to be required. Since $i$ could range through the whole set $I$, we have arrived at the desired result.

Theorem 2.8: Every $c$ system in the state space of a commutative $C^*$ algebra with unit is generated by a uniquely determined stochastic dynamic. In case of a commutative $W^*$ algebra and a normal $c$ system a stochastic dynamic in $ST(\Lambda)$ is uniquely given.

**Proof:** Let $\{\omega_s \} = \omega_{S,\beta}$ be the $c$ system in question. Then, $\omega_s, \omega_t, s \leq t$ whenever $t \geq s \geq 0$, thus Lemma 2.7 applies to the time cuts $\omega_s, \omega_t$ whenever $s \leq t$. Thus $ST(\Lambda)$ is uniquely determined.

That is we have $T \in ST(\Lambda)$ with $\omega_t = T \omega_s$ for any $\omega_s \in S_\beta$. Since $A^*$ is dense in $A^*$ and $T$ is bounded, there is no other bounded linear map performing the transition from $s$-cut to $t$-cut. Hence we may define $T = T$. This can be made for every pair with $t \geq s \geq 0$, and since $\omega_t = T \omega_s$ has to hold. Let $t \geq s \geq 0$. Then, $\omega_t = T \omega_s$ and, by the same reasoning as above, we necessarily have $T_{s,\beta} \omega_s = T \omega_s$, and $T_{s,\beta}(A^*)$ dense in $A^*$ by triviality follows. Finally, in case of a $W^*$ algebra and a normal $c$ system the assertion follows from the fact that $A^*$ is a Banach space and the above constructed stochastic maps have a dense set of $A^*$ into $A^*$, so the restriction to $A^*$ is in $ST(\Lambda)$.

To make the correspondence between $c$ systems and stochastic dynamics complete, let us note that due to (2.3) any system of trajectories $\omega_s = T \omega_s$ with $\omega_s$ running through a set $S_\beta$ with $S_\beta \subset S_\beta$ being dense in $A^*$ yields a $c$ system $[2,4$ corresponds to $S_\beta \subset S_\beta]$. Let $Z = \{\omega_s \} | s \in \Lambda \in S_\beta$ be a $c$ system in $S_\beta$. $Z$ is said to be a continuous $c$ system if any trajectory is continuously depending on $t$ at any instant $t > 0$. $Z$ is said to be differentiable if the time derivative $d_t \omega_s$ exists at any instant (at $0$ the right derivative). Clearly, differentiable $c$ systems will deserve our main interest, for the state solutions of many important master equation number among them (cf. Example 2.5).

We will justify the subsequent formulated regularity properties for continuous and differentiable $c$ systems, respectively:

**Proposition 2.9:** Let $Z$ be a continuous $c$ system in $S_\Lambda$. Then, the $Z$ generating stochastic dynamics $(T_{s,\beta})$ is strongly continuous, i.e.,

\begin{equation}
T_{s,\beta} = \lim_{t \rightarrow s} T_{t,\beta} \omega, \ \forall \omega \in A^*, \ t > s.
\end{equation}

**Proof:** It is plain to see that $lim_{t \rightarrow s} T_{t,\beta} \omega = T_{s,\beta} \omega$, $\forall \omega \in A^*$, $\forall t \geq s > 0$, for, the relation holds on a dense subset $A^* = A^*$ and $(T_{s,\beta})$ is uniformly bounded there. On the other hand, for $s > t > 0$, $s' > t' > 0$ we have

\begin{equation}
\| T_{s,\beta} \omega_t - T_{s,\beta} \omega_t \| < \| T_{s,\beta} - T_{s,\beta} \| \omega_t < 0
\end{equation}

and since $T_{s,\beta} \omega_t = T_{s,\beta} \omega_t$, by the uniform boundedness of $(T_{s,\beta})$ it follows from (2.11) that $\lim_{t \rightarrow s} T_{s,\beta} \omega_t = T \omega_t$ and, taking into account the denseness of $A^*$ in $A^*$, and stressing again the uniform boundedness argument, we see that $\lim_{t \rightarrow s} T_{s,\beta} \omega_t = T \omega_t$ for any $\omega \in A^*$.

Finally, because $T_{s,\beta} = T_{s,\beta} T_{s,\beta}$, for every $u$ with $s' > \beta > s'$, we get from the proven separate continuity, and uniform boundedness once more inserted, that with $s' > \beta > s'$

\begin{equation}
\lim_{t \rightarrow s} T_{s,\beta} \omega_t = T \omega_t = T_{s,\beta} \omega_t .
\end{equation}

and for $s = t$ we may use (2.11) to make the argument complete.

Q.E.D.

For a differentiable $c$ system $Z$ we have to state the following fact:

**Proposition 2.10:** Let $Z$ be a differentiable $c$ system in $S_\Lambda$. Then, there is a family $(L_s)_{s \geq 0}$ of linear operators, each of which is densely defined in $A^*$, such that Cauchy problem of the master equation

\begin{equation}
d_t \omega_t = L_s \omega_t ,
\end{equation}

has a solution for the dense set $[S_\beta \subset S_\beta]$ of initial elements, and the trajectories of the $c$-system $Z$ are among the state solutions of (2.12). Moreover, a solution of (2.12) starting out from a state contained in $S_\beta$ evolves in $S_\beta$ exclusively.

**Proof:** By Theorem 2.8 we are assured of a stochastic dynamic $(T_{s,\beta})$ with

\begin{equation}
\omega_t = T_{s,\beta} \omega, \ \omega \in S_\beta.
\end{equation}

Since $Z$ is differentiable, we get from (2.13)

\begin{equation}
d_t \omega_t |_{s,\beta} = \lim_{t \rightarrow s} (t - s) (T_{s,\beta} - t) \omega_t , \ \omega \in S_\beta.
\end{equation}

On the dense linear set $A^*$ we define an operator $L_s$ acting into $A^*$ by

\begin{equation}
\omega = \sum_{t} r_t \omega_t |_{s,\beta} = L_s \omega = \sum_{t} r_t d_t \omega_t |_{s,\beta}.
\end{equation}

Then, due to (2.13) and (2.14), we can be sure that $L_s$ is linearly well defined on $A^*$. Let $\omega \in S_\beta$ be a state. Then, $\omega = \sum_{t} r_t \omega_t |_{s,\beta}$ with some real $r_t$ and states $\omega_t \in \omega$. Since $T_r \omega_t \in ST(\Lambda)$, we have

\begin{equation}
\omega = T \omega_t \omega_t \in S_\beta,
\end{equation}

with $\omega = \sum_{t} r_t \omega_t$. Because of (2.14), however, we see $d_t \omega_t |_{s,\beta} = \sum_{t} r_t d_t \omega_t |_{s,\beta} = \sum_{t} r_t L_s \omega_t = L_s \omega_t$, i.e.,

\begin{equation}
\omega_t |_{s,\beta} \omega \in S_\beta \cap S_\beta \text{ is a solution of the Cauchy prob-}
\end{equation}


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lem for \( d_s \varphi = L_s \varphi \) evolving totally in \( S_A \) [due to (2.16)]. That the problem has a solution for any \( \varphi \in [S_A] \) is seen in the same way.

Q.E.D.

In other words, any differentiable \( c \)-system can be interpreted as a subset of solutions of the Cauchy problem for a suitable master equation that is solvable on a dense set of initial conditions such that the equation admits the trajectory of an initial state to evolve in the state space.

Remark 2.11: In one case of a commutative \( W^* \)-algebra \( A \), all the derived results of this part remain true statements if we make the following replacements:

- \( A^* \) replaced with \( A_* \),
- \( S_A \) replaced with normal states,
- (proper) \( c \)-system replaced with (proper) normal \( c \) system,
- \( ST(A) \) replaced with \( ST_* (A) \),
- stochastic dynamic replaced with normal stochastic dynamic,

etc.

3. \( c \) SYSTEMS AND MASTER EQUATIONS

The aim of this part is to clarify the structure of those master equations on \( A^* \) that admit proper \( c \)-systems as state solutions. We start with a class of master equations which are quite regular from our point of view.

Theorem 3.1: Let \( [L_t]_{t \in \mathbb{R}} \) be a family of bounded linear operators on \( A^* \). Assume the following conditions to hold:

\[
\left\| L_t \varphi - L_{t_1} \varphi_{t_1} \right\| \leq \varepsilon, \quad \left\| \varphi_t - \varphi_{t_1} \right\| \leq \varepsilon, \quad \forall \varphi \in [S^*_A], \quad \forall t \in [t_{i-1}, t_i],
\]

\[
(1 + e^{C(t-s)}) \left\| \varphi_t \right\| \left\| \beta^t \right\| \leq \varepsilon,
\]

Then, from (3.7) it arises

\[
\left\| \int_t^{t_1} L_t \varphi_t \omega \frac{d}{dx} - \frac{(t-s)}{N} \sum_{k=1}^N L_k \varphi_{t_1} \right\| \leq 2 \varepsilon (t-s),
\]

whenever \( u \in \{t_{i-1}, t_i\} \). Let us define \( \varphi^t = \varphi_t, \quad r \in [s, t] \),

\[
\varphi^t = \frac{(t-s)}{N} \sum_{k=1}^N L_k \varphi_{t_1}
\]

and inductively

\[
\varphi^t = \varphi^s \left( \frac{(t-s)}{N} \sum_{k=1}^N L_k \varphi^t \right) \text{ for } r \in [t_{i-1}, t_i].
\]

One easily checks that (3.8)–(3.11) guarantee that

\[
\left\| \varphi^t - \varphi^s \right\| \leq 2 \varepsilon (t-s) \beta^n,
\]

and in summing up over \( n \), running from 1 to \( N \)

\[
\left\| \varphi^t - \varphi^s \right\| \leq 2 \varepsilon (t-s) \beta^n \quad \forall \varphi \in [s, t].
\]

In using (3.10) and (3.11) we will also obtain a representation of \( \varphi^t \) in the following form

\[
\varphi^t = \prod_{k=1}^N \left( \frac{t-s}{N} \right) L_k \varphi - D_{t,s} \varphi,
\]

with

\[
D_t = \frac{(t-s)}{N^2} \left( \sum_{i=1}^N L_{i_1} \sum_{j=1}^N L_{i_2} \cdots \sum_{t_1} L_{t_1} \varphi_{t_1} \right) - L_{t_1} \cdots L_{t_N} \varphi_{t_N}.
\]

From (3.6) it arises that \( \left\| \varphi_t \right\| \leq e^{C(t-s)} \left\| \varphi \right\|, \quad \forall \varphi \in [s, t] \)

hence we may estimate the norm of \( D_t \) given by (3.15) as

\[
\left\| D_t \right\| \leq \beta^n \left\| \left( e^{C(t-s)} + 1 \right) \right\| \leq \varepsilon,
\]

where we made use of (3.8).

We define

\[
T_t = \left( \int_{s'}^{s'} \frac{(t-s)}{N} L_{s'} \right) \cdots \left( \int_{s'}^{s'} \frac{(t-s)}{N} L_{s'} \right) \cdots L_{s'} \varphi_{s'},
\]

and from (3.13) and (3.14) comes that
\[\|\varphi_j - T^*_t \varphi_l\| \leq \left(1 + \frac{2(t-s)}{(1-\beta)}\right) \epsilon,\]

and since \(\epsilon > 0\) and \(\varphi \in A^*\) were arbitrarily chosen, we may take as proven

\[T_{ts} = \text{st}-\lim_{s \to t} T^*_t \ (\text{st}-\text{strong}),\]

with \(T^*_t\) defined as above (where \(t_i = (i/N)(t-s)+s\)).

Because of (3.18) and the special structure of \(T^*_t\), we see that \(T^*_t \in \mathcal{S}(A)\) for \(N > (t-s)G(t,s)\), so there is, for fixed, \(T_{ts} \in \mathcal{S}(A)\) due to weak compactness of \(\mathcal{S}(A)\). Moreover, for \((t-s)C < 1/2\) we see

\[\|id - T^*_t\| \leq \frac{1-t-s}{1-(t-s)C} < 1, \ \forall \ N,\]

hence \(\|(id - T^*_t)\| < 1\), too. The latter means invertibility of \(T_{ts}\) in \(B(A^*)\). Since \(T_{ts}T^*_t = T^*_t\) holds, \(T^*_t\) is stochastic and bounded invertible for every pair \(t, s\), i.e., the state solutions of (3.4) form a proper \(c\) system.

Q.E.D.

We remark that Theorem 3.1 is a statement which closely relates us to Corollary 2.2.

Let us close our considerations in proving that the class of equations described in Theorem 3.1 is primary in the set of all master equations that admit a proper \(c\) system as a solution

Theorem 3.2: Let \(\{L_t\}_{t \geq 0}\) be a family of linear operators on \(A^*\) such that the Cauchy problem for

\[d_t \varphi = L_t \varphi\]

admits state solutions that form a proper \(c\)-system.

Then, every \(L_t\) is bounded and there exists a sequence of families \(\{L_t^*_t\}_{t \geq 0}\) of bounded linear operators, each of them being of the type described in Theorem 3.1 such that

\[L_t = \text{st}-\lim_{s \to t} L^*_t, \ \forall \ t \geq 0.\]

Q.E.D.

Proof: Let \(\{(\omega_i)_t\}_{t \geq 0}\) denote the proper \(c\) system known to be a solution of the master equation under discussion (the choice of \(S_{ts}/S_t\) is not a restriction). By Theorem 3.2, we are assured of the existence of a generating dynamic \(T_{ts}\) which, due to Proposition 3.3, is strongly continuous since \((\omega_i)_{t \geq 0}\) is a solution of a master equation. We define

\[L^*_t = \sigma(T_{ts} - id)\]

Inserting \(\omega_t\), we see from (3.20) that the assumptions

\[\lim_{t \to s} L^*_t \omega_t = \lim_{t \to s} \frac{\omega_t - \omega_s}{t-s} = d_t \omega_t \ | L_t \omega_t, \ \forall \ \omega \in A^*.\]

(3.21)

Since (3.21) is valid on \((\omega_t)_{t \geq 0}\), it is valid on \(\{(\omega_t)_{t \geq 0}\} = A^\ast\), too, i.e., \(\lim_{t \to s} L^*_t \omega = L_t \omega, \ \forall \ \omega \in A^\ast\). The principle of uniform boundedness (Banach–Steinhaus theorem) then gives that \(L_t\) has been bounded for every \(t \geq 0\). Strong continuity of \((T_{ts})\) implies \(L^*_t\) to be strongly continuous. Finally, the special form of (3.20) makes all the other requirements of Theorem 3.1 (3.1)-(3.3) hold for \(\{L^*_t\}_{t \geq 0}\).

Q.E.D.
\textbf{f(0, \ldots, 0) = 0, gives no contradiction in transition from (A3) to (A4). Because of } \cup_s G_s' \supseteq \Omega \text{ and since } G_s' \cap G_t' = \emptyset \text{ for } s \neq t, \text{ from (A4) it follows that}
\begin{equation}
\sum f(x_t, \ldots, x_s) = \sum \mu(G_s) f(x_t, \ldots, x_s),
\end{equation}
and the right-hand side of (A5) equals \( \int \omega(x_1, \ldots, \omega(x_s)) \, d\mu(x) \). We may suppose that \( \omega_1 \) corresponds to a finite orthogonal decomposition of 1 into orthoprojections \([x_i]_1 \text{ of } \Omega \text{ has to vanish, so we see from (A5)}
\begin{equation}
\sup \{ \omega_1 \} \sum f(x_1, \ldots, \omega(x_s))
= \int \omega(x_1, \ldots, \omega(x_s)) \, d\mu(x),
\end{equation}
where the sup runs through all finite orthogonal decompositions of the unity. The left-hand side of (A6), however, equals \( S_f(\omega_1, \ldots, \omega_s) \) by the results on \( h \)-convex functionals \([\text{see (5.9 in Ref. 1)}], \text{i.e., we may take as proven}
\begin{equation}
S_f(\sigma_1, \ldots, \sigma_s) = \int \omega(x_1, \ldots, \omega(x_s)) \, d\mu(x)
\end{equation}
where \( \omega(x_1, \ldots, \omega(x_s)) \text{ corresponds to normal states on } L^1(\Omega, \mu). \text{ Then, the continuity of } f \text{ on } L^1(\Omega, \mu) \text{ is measurable, and positive by assumption. First, let us show the existence of an increasing sequence } \Omega_n \text{ of measurable sets with } \mu(\Omega_n) < \infty, \omega(x_1, \ldots, \omega(x_s)) \in L^1(\Omega, \mu), \forall x, \text{ and}
\begin{equation}
\lim_{n \to \infty} \int \omega(x_1, \ldots, \omega(x_s)) \, d\mu(x)
= \int \omega(x_1, \ldots, \omega(x_s)) \, d\mu(x),
\end{equation}
where \( \omega(x) \text{ stands for the characteristic function of } \Omega \text{. In fact, let us define } \Omega_n = \{ x \in \Omega : (1/n) < h(x) < n \}, \text{ with } h(x) = \sum_i \omega_i(x) \text{. Then, } \Omega_n \subseteq \Omega_n \subseteq \cdots \text{ and } \mu(\Omega_n) < \infty \text{ since } h(x) \in L^1(\Omega, \mu) \text{. Setting } \Omega' = \cup_n \Omega_n \text{, we see}
\int f(x_1, \ldots, \omega(x_s)) \, d\mu(x) = \int f(x_1, \ldots, \omega(x_s)) \times d\mu(x), \text{ for, from } x \notin \Omega' \text{ it follows that either } h(x) = 0 \text{ by homogeneity, or } h(x) = \infty, \text{ which happens at most on a set of measure zero } [h \in L^1(\Omega, \mu)]. \text{ Thus, by Lebesgue's monotone convergence theorem we see equality (A5) to be true (f is positive).}
We are going to show that
\begin{equation}
\int \omega(x_1, \ldots, \omega(x_s)) \, d\mu(x) \leq S_f(\sigma_1, \ldots, \sigma_s),
\end{equation}
which consist of simple functions with support in \( \Omega_n \text{ such that}
\begin{equation}
\omega_{x_1} \geq \omega_{x_1} \geq \omega_{x_1} \forall i, \omega_{x_1} = \sum f(x_1, \ldots, x_s) \text{ (with } x_i \neq 0),
\end{equation}\text{ and from the second part of (A10) and from (A12) we see}
\begin{equation}
\lim E^x \omega \omega = E^x \omega \omega \text{ uniformly } \forall i.
\end{equation}
Because the adjoint map of } E^x, E^x, \text{ is positive and } E^{x+1} = x \leq 1, \text{ from this, together with positivity of } f \text{ and the original definition of } S_f \text{ [cf. (1.1)]}, \text{ follows}
\begin{equation}
S_f(\omega_1, \ldots, \omega_s) \leq S_f(\omega_1, \ldots, \omega_s)
\end{equation}
The \( E^x \omega \) being simple functions makes (A7) to be applicable, thus from (A11) we are led to
\begin{equation}
\int \int f(x_1, \ldots, \omega(x_s)) \, d\mu(x) \leq S_f(\omega_1, \ldots, \omega_s),
\end{equation}
Applying (A13) and recalling \( \mu(\Omega_n) < \infty, \text{ (A15) yields}
\begin{equation}
\int \int f(x_1, \ldots, \omega(x_s)) \, d\mu(x) \leq S_f(\omega_1, \ldots, \omega_s),
\end{equation}
from which inequality by means of (A9) the desired result (A9) can be seen.
Let us demonstrate the validity of the reverse of the inequality just proved. To this sake, by standard methods (see Ref. 4) we construct increasing sequences \( \{ s_{x_1}, \ldots, \omega_s \} \) of measurable simple functions with \( 0 \leq s_{x_1} \leq s_{x_2} \leq \cdots \leq \omega_s \) and \( \lim_{n \to \infty} S_f(\sigma_n(x_1, \ldots, \omega_s)) = \omega(x) \text{ for all } \forall x \in \Omega \text{. We can choose the sequences in such a way that the convergence is uniform on any subset of } \Omega \text{ where } \omega_s \text{'s are bounded.}
Especially, \( \{ s_{x_1, \ldots, x_s} \} \) tends uniformly to \( \omega_s \text{ on } \Omega_n \text{ as defined above. One also easily recognizes that } \int s_{x_1, \ldots, x_s} \times d\mu(x) = 1. \text{ Now, by definition of } S_f \text{, we have}
\begin{equation}
S_f(\sigma_1, \ldots, \sigma_s) = S_f(\sigma_1, \ldots, \sigma_s),
\end{equation}
with
\begin{equation}
S_f(\sigma_1, \ldots, \sigma_s) = \sup \sum f(\sigma_1, \ldots, \omega_s(\sigma_s)),
\end{equation}
with the sup running through all positive decompositions of 1 into at most, \( M \) positive elements. It is plain to see that \( S_f \) is \( \| \cdot \| \) -continuous, so from \( \| S_f(x) - \omega_s \| = 0 \) follows that
\[ S^f_T(\omega_1 x_1, \ldots, \omega_n x_n) = \lim_{n \to \infty} S^f_T(s_{1,n} x_1, \ldots, s_{n,n} x_n) \]
\[ = \lim_{n \to \infty} \int f(x_{s_{1,n}}, \ldots, x_{s_{n,n}}) \, d\mu(x) \]
\[ = \int f(\omega_1(x), \ldots, \omega_n(x)) \, d\mu(x) \]
\[ = \int f(\omega_1(x), \ldots, \omega_n(x)) \, d\mu(x) \]

where in the last steps we made use of (A7) and the uniform convergence of \( \{s_{i,n}\} \) towards \( \omega_i \) on \( \Omega \). With finite measure, and positivity of \( f \) makes the conclusion of (A18) complete. Since the increasing sequences \( \{\omega_i x_n\}_n \) converge pointwise to \( \omega_i \) on \( \Omega' \), from the property of \( \Omega' \) (see above) and positivity of \( \omega_i \) comes that \( \|\omega_i x_n - \omega_i \| \to 0 \). Hence, from \( \|\cdot\|_1 \)-continuity of \( S^f_T \) comes that (A18) can be turned into

\[ S^f_T(\omega_1, \ldots, \omega_n) = \int f(\omega_1(x), \ldots, \omega_n(x)) \, d\mu(x) \]  

and in applying (A17) to (A19) we have arrived at

\[ S_f(\omega_1, \ldots, \omega_n) = \int f(\omega_1(x), \ldots, \omega_n(x)) \, d\mu(x) \]  

Taking together (A20) with (A9), the desired result (A1) is obtained. We remark that the value \( = \) is included in all considerations.

Q.E.D.