Quantization method using Dirichlet's principle

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1. In the following we try to approach quantization problems in given curved space-time manifolds by some ideas of Euclidean quantum field theory /1/ as proposed by Nelson, Dobruschin, Guerra, Rosen, Simon,... We use a reformulation /2/ coming from the study of reflexion positivity advocated by Hegerfeld, Fröhlich, Lieb,... The advantage of "Euclidean-Riemannian" methods is in obtaining suitable "1-particle" Hilbert spaces attached to space like hypersurfaces, and in getting positive definite Hamiltonians acting on that hypersurface depending Hilbert spaces and governing the time evolution in the Tomonaga Schwinger sense.

2. Let us consider a 4-manifold, \( M \), equipped with a metrical tensor, \( g_{ik} \), of signature (++++).
Let \( e_i \) denote a time-like and normalized vector-field defined on \( M \).
This setting introduces uniquely a Riemann metric by
\[
g_{ik} + \tilde{g}_{ik} = 2 e_i e_k
\]
and we consider the Laplace-Beltrami operator
\[
\tilde{\Delta} \quad \text{belonging to} \quad \tilde{g}_{ik}
\]
Becoming an elliptic differential operator there exists for every open subset, \( \mathcal{D} \), of \( M \), and for every regular function, \( f \), the support of which is contained in \( \mathcal{D} \), and for given "rest mass", \( m \), one and only one regular function \( h \) on \( \mathcal{D} \) satisfying
(a) \( (-\tilde{\Delta} + m^2) h = f \)
(b) For every \( \varepsilon > 0 \) there is a compact \( K \) within \( \mathcal{D} \) such that
\[
|h(x)| \leq \varepsilon \quad \text{if} \quad x \in \mathcal{D} \quad \text{and} \quad x \notin K.
\]
Clearly, \( h \) is the solution of (a) in \( \mathcal{D} \) fulfilling the 
Dirichlet boundary condition. We denote it by
\[
h = G_D f
\]
which in turn defines the appropriate Green's operator \( G_D \).
Later on we heavily use Dirichlets' principle.
Let $h'$ be another solution of (a),

$$(-\tilde{\Delta} + m^2)h' = f$$

beeing bounded together with its first derivatives globally on $D$. (Remind $h'$ is regular automatically.)

Then

$$\int_D h' f \tilde{dv} \geq \int_D h f \tilde{dv} \geq 0$$

and the equality sign is valid in the first inequality iff $h' = h$, and in the second iff $f = 0$.

The Dirichlet energy integral will be the starting point for the construction of scaler product, while the Dirichlet principle shall guaranty the positive definiteness of Hamilton operators.

3. We now define the Hilbert spaces $N_D$ as the completion of the pre-hilbertian space $N_D^0$ consisting of all distributions $f$, with compact support within $D$, for which the Dirichlet energy integral

$$(c) \quad \langle f, f \rangle_D := \int_D \left\{ \tilde{v}^i \left( \partial_i G_D f \right)(\partial_i G_D f) + m^2 \left( a_D \tilde{f} g_D f \right) \right\} \tilde{dv}$$

exists. (c) defines, by polarization, the scalar product of $N_D$ uniquely.

The set $N_D^0$ (not its scalar product) is independent of the choice of the time-like normalized vector field $e_i$.

If furthermore $D_1 \subset D_2$, then $N_{D_1}^0 \subset N_{D_2}^0$ naturally. The Dirichlet principle assures

$$(d) \quad \langle f, f \rangle_{D_2} \geq \langle f, f \rangle_{D_1} \quad \text{if} \quad f \in N_{D_1}^0$$

and both norms are coordinated.

Hence there is a natural contraction

$N_{D_1} \longrightarrow N_{D_2} \quad \text{if} \quad D_1 \subset D_2$

Therefore, if $D = M$ we get the largest in form sense possible scalar product, and likewise the effect of a possible enlargement of $M$ can be seen from this.

Let us restrict ourselves to $D = M$.

$N_M$ contains "Physical 1-particle spaces": Let $\sigma$ be a (piece of a) space-like hypersurface which intersects the field $e_\downarrow$ normally.

Then we define the subspaces of $N_M$

$$N_{\sigma} := \left\{ f \in N_M \mid \text{support } f \subset \sigma \right\}$$
to be the 1-particle Hilbert space associated to particles sitting at the instant $\sigma$ at $\sigma'$. The normality of $e_i$ on $\sigma'$ is crucial for the interpretation.

There is the following nice fact: Let $\mu$ be a regular function defined on $\sigma$, let $\delta_{\sigma}$ denote the Dirac distribution concentrated on $\sigma$. Then $\mu \delta_{\sigma}$ is in $N_D$ and these functions form a dense set within $N_D$. Therefore, any 1-particle state on $\sigma'$ is uniquely given by a locally integrable function on the hypersurface in question.

4. We shall define the Hamiltonian $H_{\sigma}$. In Euclidean quantum field theory one constructs $\exp(-tH)f$, $f \in N_{\sigma}$, by first translating $f$ the time $t$ into the future and then by projecting within $N_M$ the result of the translation into $N_{\sigma}$ back.

There are more or less obvious generalization if $M$, $e_{ik}$, is time oriented.

We consider the set $T^-_{\sigma}$ of all world points of the past of the hypersurface $\sigma$. If $\sigma'$ is in the future of $\sigma$ then there is a natural contraction induced by (d):

$$N_{T^-_{\sigma}} \longrightarrow N_{T^-_{\sigma'}}$$

and hence a natural contraction of $N_{\sigma}$ into $N_{\sigma'}$.

Now, for we cannot expect to get a semigroup we try to define a generator of the time-movement depending on $\sigma$. This, indeed, is Tomonaga–Schwinger's idea. It should be done by a limiting process

$$\langle f, H_{\sigma} g \rangle_M := \lim d^{-1}(\langle f, g' \rangle_{T^-_{\sigma'}} - \langle f, g \rangle_{T^-_{\sigma}})$$

where $d$ denotes a "distance" and $g'$ a translated $g$ with $f, g \in N_{\sigma}$.

By Dirichlet's principle one automatically gets, provided the limit exists,

$$\langle f, H_{\sigma} g \rangle_M \geq 0$$

i.e. the usually extremely difficult to fulfil spectrality condition for the Hamilton operator is guarantied.

The idea is to translate $\sigma$ "infinitesimally" and geodesically ("freely falling") into the future.

It is, however, not straightforward to do this correctly.

In the case, $\sigma$ devides $M$ into two parts and if there is an isometrical refelction of $M$ with respect of $\sigma'$ and $g_{ik}$, then one may infer

$$\langle f, H_{\sigma} g \rangle_M = \int \bar{f} \ e^{i \cdot \varphi_{\sigma}} G_M \ f \ \bar{g} \ d\nu_{\sigma}$$

provided $f \in N_{\sigma}$ (see /2/).
The Dirichlet principle even gives some hints how to overcome the dependence on the vector field \( e_1 \).

We have seen \( \langle f, f \rangle_D \) getting larger for increasing \( D \).

In Minkowski space one has to choose \( e_1 \) geodesically for flat hypersurfaces. Because of time-dilatation these fields give within the contraction \( g_{ik} \longrightarrow \tilde{g}_{ik} \) the "longest" trajectories.

Hence, possibly, the Dirichlet integral sees \( M \) "largest" if \( e_1 \) is as near as possible to a geodesic normalized field.

Hence, heuristically, one had to define

\[
\langle f, f \rangle_{\sigma} = \sup \langle f, f \rangle_M
\]

where the supremum has to run through all the possibilities to make the constructions above, i.e., it has to run through all normalized time-like vector fields cutting \( \sigma \) normally. (Remind that our scalar products depend on \( e_1 \).)

Then, generally, the supremum would not necessarily provide us with a new Hilbert norm but only with a Banach norm, and then the 1-particle states would be not Hilbert but only Banach spaces.

Very speculatively this may indicate the "instability" of the 1-particle states. (In view of second quantization the vacuum "instability" had to imply the "instability" of the 1-particle states and vice versa.)

Unfortunately, even in Minkowski space I have been unable to prove or disprove that the supremum above will be attained by the time-like geodesics. This, however, is a necessary requisit to consider the discussions of this last section seriously.

References:

/1/ B. Simon: The \( P_2 \) Euclidean (Quantum) Field Theory, 1974, Princeton University Press.