THE "TRANSITION PROBABILITY" IN THE STATE SPACE OF A *-ALGEBRA

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Let ω, ϱ be two states of a *-algebra and let us consider representations of this algebra R for which ω and ϱ are realized as vector states by vectors x and y. The transition probability $P(\omega, \varrho)$ is the supremum of all the numbers $|(x, y)|^2$ taken over all such realizations. We derive properties of this straightforward generalization of the quantum mechanical transition probability and give, in some important cases, an explicit expression for this quantity.

1. Introduction

In this paper we consider an expression $P(\omega_1, \omega_2)$, which we shall call the *transition probability* between two states ω_1, ω_2 of a given *-algebra. This name is reasonable especially for pure states: For normal pure states of a type I von Neumann algebra P is what is usually called the "transition probability" in quantum theory. However, a correct physical interpretation in the general case of mixed states is not known, though this quantity appears quite naturally in the so-called algebraic approach.

The expression P, which we are going to define, was already considered by Kakutani [3] for Abelian and by Bures [2] for general W^*-algebras and used by these authors in the construction of infinite tensor products.

The aim of the present paper is to show the concavity of P, to establish the connection of P with support properties of states (i.e. their orthogonality), and, using an idea of Araki [1], to calculate P in some important examples.

If, for instance, the two states are given by the density matrices d_1 and d_2 (with respect to a type I factor), then

$$P = (\mathrm{Sp} \cdot s)^2, \quad s = (d_1^{1/2} d_2 d_1^{1/2})^{1/2}. \tag{1}$$

This rather complicated expression reduces simply to

$$|(x, y)|^2 \tag{2}$$

if the density matrices represent pure states that are given by the normed vectors x, y of the underlying Hilbert space.

2. Definition and some properties of P

Let us denote by R a *-algebra with unit element e. The simple idea underlying the definition of $P(\omega_1, \omega_2)$ is the following: Consider a *-representation π of R and suppose that there are vectors x_1, x_2 in its domain of definition D_π, which induce the states ω_1, ω_2, i.e. for all $b \in R$

$$\omega_j(b) = (x_j, \pi(b)x_j). \tag{3}$$

The number $|(x_1, x_2)|^2$ then depends on the representation π and the choice of the vectors x_1, x_2 in D_π. We then define [2] accordingly $P(\omega_1, \omega_2)$ to be the supremum of all numbers $|(x_1, x_2)|^2$ for which (3) is valid. Hence

$$P(\omega_1, \omega_2) = \sup|(x_1, x_2)|^2, \tag{4}$$

and the supremum runs over all *-representations π for which there are pairs of vectors x_1, x_2 satisfying (3) and over all such pairs x_1, x_2. To express its dependence on R, we sometimes write

$$P(R|\omega_1, \omega_2)$$

for the quantity (4).

From the definition one immediately gets the relations

$$0 \leqslant P(\omega_1, \omega_2) \leqslant 1, \tag{5}$$

$$P(\omega_1, \omega_2) = P(\omega_2, \omega_1), \tag{6}$$

$$P(\omega, \omega) = 1 \tag{7}$$

for all states of a given *-algebra R.

Next we prove the concavity of P with respect to Gibbsian mixtures, i.e. we prove formula (8) below.

Let us consider three states $\omega, \omega_1, \omega_2$ and two *-representations π_1, π_2 such that ω, ω_1 may be represented with the help of π_1 by the vectors x_1, y_1 and, similarly, ω, ω_2 by the vectors x_2, y_2 in the representation π_2. We are allowed to assume

$$P(\omega, \omega_j) \leqslant |(x_j, y_j)|^2 + \varepsilon.$$

In the direct sum $\pi_1 + \pi_2$ the state ω is representable by every vector $x = \lambda_1 x_1 + \lambda_2 x_2$, $|\lambda_1|^2 + |\lambda_2|^2 = 1$. On the other hand, the state $\hat{\omega} = p_1\omega_1 + p_2\omega_2$, where $p_1 + p_2 = 1$ and $p_j \geqslant 0$ is given in $\pi_1 + \pi_2$ by every vector $y = \mu_1 y_1 + \mu_2 y_2$, $|\mu_j|^2 = p_j$. Hence

$$P(\omega, \hat{\omega}) \geqslant |(x, y)|^2 = |\lambda_1 \mu_1 (x_1, y_1) + \lambda_2 \mu_2 (x_2, y_2)|^2.$$

Taking the maximum with respect to all possible values λ_1, λ_2 on the right-hand side, we get

$$P(\omega, \hat{\omega}) \geqslant |\mu_1 (x_1, y_1)|^2 + |\mu_2 (x_2, y_2)|^2.$$

Now $\varepsilon \geqslant 0$ is arbitrarily chosen. Thus we obtain, with $p_j \geqslant 0$ and $p_1 + p_2 = 1$

$$P(\omega, p_1\omega_1 + p_2\omega_2) \geqslant p_1 P(\omega, \omega_1) + p_2 P(\omega, \omega_2). \tag{8}$$

3. Orthogonality of states

We recall (Sakai [5]) that two states of a C^*-algebra are said to be orthogonal to each other if for every decomposition of $\varrho = \omega_1 - \omega_2$ into two positive linear functionals ω_1', ω_2'

$$\varrho = \omega_1' - \omega_2'$$

one has

$$\omega_1'(e) + \omega_2'(e) \geqslant \omega_1(e) + \omega_2(e) = 2.$$

Assuming now R to be an arbitrary *-algebra with identity e, we define

$$\|\varrho\| = \sup\{\omega_1'(e) + \omega_2'(e)\}, \tag{9}$$

where the supremum runs over all decompositions

$$\varrho = \omega_1' - \omega_2', \quad \omega_j' \text{ positive}. \tag{10}$$

If there is no such decomposition (10), one writes $\|\varrho\| = \infty$.

In the above-mentioned case of a C^*-algebra, two states ω_1, ω_2 are orthogonal one to another iff $\|\omega_1 - \omega_2\| = 2$. We show that for all *-algebras with identity the relation $\|\omega_1 - \omega_2\| = 2$ implies $P(\omega_1, \omega_2) = 0$. This follows from an inequality, which we are now going to prove and which reads for any two states

$$\|\omega_1 - \omega_2\| \leqslant 2\sqrt{1 - P(\omega_1, \omega_2)}. \tag{11}$$

Let us assume that ω_1, ω_2 are represented as vector states by the vectors x_1, x_2 of a given *-representation π of R. With the help of the one-dimensional projectors q_j determined by x_1, x_2 the functional $\varrho = \omega_1 - \omega_2$ is given by

$$\varrho(a) = \mathrm{Sp.}\{(q_1 - q_2)\pi(a)\}. \tag{12}$$

There are projection operators $\bar{q}_j$ satisfying $\bar{q}_1 \cdot \bar{q}_2 = 0$ and

$$q_1 - q_2 = \lambda_1 \bar{q}_1 - \lambda_2 \bar{q}_2. \tag{13}$$

It follows that $\omega_1 - \omega_2 = \omega_1' - \omega_2'$, where

$$\omega_j'(a) = \lambda_j \mathrm{Sp.}\{\bar{q}_j \pi(a)\}.$$

Hence

$$\|\omega_1 - \omega_2\| \leqslant \lambda_1 + \lambda_2. \tag{14}$$

Now we take the trace in (13) and obtain $\lambda_1 = \lambda_2 = \lambda$. Squaring (13), we get

$$q_1 + q_2 - q_1 q_2 - q_2 q_1 = \lambda^2(\bar{q}_1 + \bar{q}_2).$$

Taking the trace, one obtains

$$2 - 2|(x_1, x_2)|^2 = 2\lambda^2.$$

Because of (14) we therefore conclude that

$$\|\omega_1 - \omega_2\| \leqslant 2\sqrt{1 - |(x_1, x_2)|^2}.$$

If π runs over all *-representations, we obtain the inequality (11) and the assertion is proved.

4. An estimation from above

The transition probability is defined as a supremum. Therefore it is interesting to have an estimation of P from above. To derive such an inequality we use the notation of the "geometrical mean" of two positive Hermitian forms introduced by Woronowicz. Let $\beta_1(x, y)$, $\beta_2(x, y)$ denote two positive semidefinite Hermitian forms on some linear space L. Then, according to Pusz and Woronowicz [4], there exists on L one and only one form $\beta(x, y)$ with the properties:

(i) $|\beta(x, y)|^2 \leqslant \beta_1(x, x)\beta_2(y, y)$.

(ii) If $|\beta'(x, y)|^2 \leqslant \beta_1(x, x)\beta_2(y, y)$ with a positive semidefinite form β', then

$$\beta'(x, x) \leqslant \beta(x, x).$$

This Hermitian form β, uniquely determined by β_1, β_2, will be denoted by the symbol

$$\sqrt{\beta_1 \beta_2}$$

and called the *geometrical mean* of β_1 and β_2.

Let us now consider a state ω of a *-algebra R. ω defines two Hermitian forms

$$\omega^R(b, a) = \omega(ab^*), \tag{15}$$

$$\omega^L(ba) = \omega(b^*a).$$

Now the inequality in question is

$$P(\omega_1, \omega_2) \leqslant \beta(e, e), \quad \beta = \sqrt{\omega_2^R \omega_1^L}. \tag{16}$$

To prove this, we assume π to be a *-representation of R for which (3) holds. Defining now

$$\beta'(a, b) = \big(x_1, \pi(a)x_2\big)\big(x_2, \pi(b)x_1\big),$$

we get

$$\beta'(e, e) = |(x_1, x_2)|^2, \quad \beta'(a, a) \geqslant 0$$

and see that $|\beta'(a, b)|^2$ is smaller than $\omega_2(aa^*)\omega_1(b^*b)$ which already shows the validity of (16).

5. Calculation of P

Let us first mention that in all relevant cases P coincides with the usual quantum mechanical transition probability: Let x_1, x_2 be two normed vectors of a Hilbert space H. If R is an operator *-algebra of H, i.e. a *-subalgebra of some algebra $L^+(D)$, D dense in H, then the following is true (Uhlmann [6]): If

$$\omega_j(A) = (x_j, Ax_j), \quad A \in R,$$

we have

$$P(R|\omega_1, \omega_2) = |(x_1, x_2)|^2$$

if R contains the projection operators onto x_1, x_2.

However, this gives us P for pure states only. Let us now try to give an explicit expression for P allowing ω_1, ω_2 to be mixed and generalizing the above-mentioned result.

THEOREM. *Let ω_1, ω_2 be two states of the C^*-algebra R. If there exists a positive linear form ω of R and two elements $b_1, b_2 \in R$ with*

$$\omega_j(a) = \omega(b_j^* a b_j), \tag{17}$$

$$b_1^* b_2 = b_2^* b_1 \geqslant 0, \tag{18}$$

then

$$P(\omega_1, \omega_2) = \omega(b_1^* b_2)^2. \tag{19}$$

Before proving the theorem we shall first convince ourselves that it implies equation (1), first assuming that R is the algebra $B(H)$ of all bounded operators of the Hilbert space H. For this purpose we choose $\omega(a) = \mathrm{Sp}\, ad$ such that $d_j d^{-1}$ is bounded for $j = 1, 2$. Condition (17) now reads

$$d_j = b_j d b_j^* \tag{20}$$

and we have to satisfy (18). This is done by writing

$$\begin{aligned} b_1 &= d_2^{-1/2}(d_2^{1/2} d_1 d_2^{1/2})^{1/2} d^{-1/2}, \\ b_2 &= d_2^{1/2} d^{-1/2}. \end{aligned} \tag{21}$$

Here we have to define b_1 by a limiting procedure if d_2 is singular. It follows that

$$b_1^* b_2 = d^{-1/2} s d^{-1/2} \quad \text{with } s = (d_2^{1/2} d_1 d_2^{1/2})^{1/2} \tag{22}$$

and according to (19)

$$P\big(B(H)|\omega_1, \omega_2\big) = (\mathrm{Sp}\, s)^2, \tag{23}$$

i.e. formula (1).

We may extend this result considerably with the help of a simple observation. Assuming for two algebras the relation $R_1 \subseteq R_2$, we find

$$P(R_1|\hat{\omega}_1, \hat{\omega}_2) \geqslant P(R_2|\omega_1, \omega_2) \tag{24}$$

if only $\hat{\omega}_j$ are the restrictions of the states ω_j of R_2 on R_1. To see this, we have only to take into account that every *-representation of R_2 determines a representation of R_1, namely its restriction to R_1.

Applying this remark and the uniqueness of the extensions of states considered below, one can prove the following: Let D be a dense linear manifold of the Hilbert space H and let d_1, d_2 denote two normed density operators. These density operators define two states $\bar{\omega}_j$ of the algebra $L^+(D) \cap K$, where K is the *-algebra generated by the identity map and the compact operators. If now R is an operator *-algebra satisfying

$$L^+(D) \cap K \subseteq R \subseteq L^+(D) \tag{25}$$

and if we can extend the $\bar{\omega}_j$ to states ω_j of R, we get

$$P(R|\omega_1, \omega_2) = (\mathrm{Sp}\, s)^2, \tag{26}$$

where s is given by (22).

Let us now discuss a special case of the situation described above, in which ω_2 is a pure state. Then there is a vector $x \in H$ with

$$d_2 y = (x, y)y, \quad y \in H$$

and a short calculation shows that

$$P = (x, d_1 x).$$

Therefore, from

$$d_1 y = \sum \lambda_j (y_j, y) y_j, \quad y \in H$$

we obtain

$$P = \sum \lambda_j |(x, y_j)|^2,$$

which is completely reasonable and natural.

We further mention the consequences of the theorem for commutative C^*-algebras. Let $R = C(X)$ denote the algebra of continuous functions on the compact X and consider two states of R which may be represented on X by a measure dv on X and by their Radon–Nikodym derivatives h_j as

$$\omega_j(a) = \int_X a(t) h_j(t) dv. \tag{27}$$

We then get

$$P(R|\omega_1, \omega_2) = \left[\int_X \sqrt{h_1(t) h_2(t)}\; dv\right]^2. \tag{28}$$

This indicates the difficulty in interpreting P as a "transition probability" if *both* states are mixed ones.

Finally, we want to remark that from

$$b_1^* b_2 = b_2 b_1^*, \tag{29}$$

which is true for *commuting* density operators and in every *commutative* C^*-algebra, the result of the theorem can be written with the aid of geometrical means as

$$P = \left[\sqrt{\omega_1^R \omega_2^R}(e, e)\right]^2. \tag{30}$$

6. Proof of the theorem

At first we convince ourselves that (19) gives a lower bound for P. Indeed, one only has to take the GNS-construction associated with the state ω mentioned in the theorem to see this.

In the next step we consider an arbitrary *-representation and two of its vectors x_1, x_2 which allow to identify ω_1, ω_2 as vector states (3). Then the complex linear form

$$f(a) = (x_1, \pi(a) x_2) \tag{31}$$

satisfies the Schwartz–Bunyakowski inequality

$$|f(a^*b)|^2 \leqslant \omega_1(a^*a)\omega_2(b^*b). \tag{32}$$

In the last step we consider an arbitrary complex-linear functional f on R for which (32) is true. Then, if c is a positive invertible element in R, we have

$$|f(e)|^2 \leqslant \omega_1(c)\omega_2(c^{-1}). \tag{33}$$

We choose

$$c = b_2(s+\varepsilon e)^{-1}b_2^*+\varepsilon e, \qquad \varepsilon > 0.$$

Then

$$\omega_1(c) = \omega(b_1^* c b_1) = \varepsilon\omega(b_1^* b_1)+\omega\big(s(s+\varepsilon e)^{-1}s\big),$$

and we have

$$\omega_1(c) \leqslant \omega(s)+\varepsilon\omega(b_1^* b_1). \tag{34}$$

Further

$$\omega_2(c^{-1}) = \omega(k)$$

with

$$k = b_2^*\{b_2(s+\varepsilon e)^{-1}b_2^*+\varepsilon e\}^{-1}b_2.$$

If we insert in this expression $t = b_2(s+\varepsilon e)^{-1/2}$, we obtain after some straightforward calculation

$$k \leqslant \big((s+\varepsilon e)^{-1}\big)^{-1} = s+\varepsilon e.$$

Now $\varepsilon > 0$ could be chosen arbitrarily, and so we get from (33) and (34) the desired estimate

$$|f(e)|^2 \leqslant \omega(s)^2.$$

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