série des cours et conférences sur la physique des hautes énergies

N°3

AN INTRODUCTION TO THE "ALGEBRAIC APPROACH"
TO SOME PROBLEMS OF THEORICAL PHYSICS

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COURS A L'ÉCOLE INTERNATIONALE DE LA PHYSIQUE DES PARTICULES ELEMENTAIRES
BASKO POLJE, MAKARSKA
STRASBOURG-BELGRADE

1974

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_________________________ Dubna, 1974 ________________
0. Introduction

To-day practically every physicist is acquainted with the elements of the theory of groups and their applications in physics as an essential tool to handle symmetries.

Now, since several years another mathematical structure, the $\star$-algebra ("star-algebra"), enters the scenery of physics. I would not dare to bother you with these somewhat abstract things, if I were not convinced that they will become of comparative importance as groups.

In this very short introduction to that, what might be called "algebraic approach" to some problems of theoretical physics, we have grouped the material around few concepts about $\star$-algebras, their representations and states and have tried to explain some starting points for their applications. Further, we have only presented such facts, which are not restricted to the algebras of bounded operators. One or another fact may be new also to people, experienced in this field. Proofs, however, are omitted.

We have not even touched the large field of questions concerning the time development of physical systems.
We intend to introduce the concept of an algebra, or more literally, of an "associative algebra over the complex numbers". We define an algebra $\mathcal{A}$ to be a complex linear space together with a composition rule, called product, that associates uniquely with every ordered pair $a, b$ of elements of $\mathcal{A}$ another element $ab$

$$a, b \in \mathcal{A} \rightarrow ab \in \mathcal{A}$$

Furthermore, the product has to fulfill the conditions

$$\begin{align*}
(a + b)c &= ac + bc \\
c(a + b) &= ca + cb
\end{align*}$$

i.e., the distributive law, and

$$(ab)c = a(bc)$$

i.e., the associative law for every triplet $a, b, c$ of elements of $\mathcal{A}$. According to this definition we can freely add and multiply the elements of an algebra and we can also multiply them by complex numbers. In general, the multiplication is non-commutative.

If $\mathcal{A}$ denotes an algebra, a subset $\mathcal{A}_0$ is called subalgebra, if it is a complex linear subspace of $\mathcal{A}$ and if it is multiplicatively closed, i.e., from $a, b \in \mathcal{A}_0$ it follows $ab \in \mathcal{A}_0$. If we have some subalgebras, then their intersection is a subalgebra again. This simple property is often used as follows: If $\mathcal{N}$ is any subset of $\mathcal{A}$, then we call the intersection of all such subalgebras of $\mathcal{A}$ which contain the subset $\mathcal{N}$, the subalgebra generated by $\mathcal{N}$.
The algebra generated by \( N \) consists of all those elements, which can be obtained from the elements of \( N \) by repeated addition, multiplication and multiplication by complex numbers.

Let us now consider two algebras \( A \) and \( B \). A map \( \tau \) from \( A \) into \( B \) is called a homomorphism, if it "conserves" the algebraic composition rules, i.e., if and only if

\[
\begin{align*}
\tau(\lambda a + \mu b) &= \lambda \tau(a) + \mu \tau(b) \\
\tau(ab) &= \tau(a) \cdot \tau(b)
\end{align*}
\]

is valid. The set \( \tau(A) \) of all pictures of elements of \( A \) under \( \tau \) is a subalgebra of \( B \). \( \tau \) is called homomorphism onto \( B \), if \( \tau(A) = B \). If it happens that \( \tau(a_i) \cdot \tau(a_j) = 0 \), then we clearly have \( \tau(a_i - a_j) = 0 \). The set

\[
\mathcal{J} = \{ a \in A : \tau(a) = 0 \}
\]

is called the kernel of the homomorphism. Using (1.4) one finds the following properties of the kernel: (i) it is a complex linear subspace of \( A \) and (ii) it contains with an element \( a_0 \) every element \( a a_0 \) and \( a_0 a \) with arbitrary \( a \in A \). A subset of an algebra, satisfying (i) and (ii) above, is called an (two-sided) ideal. One finds that the intersection of a collection of ideals is again an ideal. Hence, as in the case of subalgebras, we may speak of the ideal generated by some subset of \( A \).

If \( \mathcal{J} \) is an ideal of \( A \), then there exists a homomorphism \( \tau \) onto an algebra \( B \) such, that the kernel of \( \tau \) coincides with the given ideal \( \mathcal{J} \). We see this from a construction that imitates the construction of
factor groups in group theory: Let us write \( \alpha_1 \equiv \alpha_2 \) for two elements of \( A \) if and only if \( \alpha_1 - \alpha_2 \in J \). In virtue of the properties (i) and (ii) of ideals, this identification of elements (or, equivalently, partition in classes modulo \( J \)) indeed gives us a new algebra, the factor algebra modulo \( J \), for which the symbol \( A/J \) is reserved.

Let us add some terminology. If the kernel of a homomorphism consists of zero only, the homomorphism is said to be an isomorphism. Two algebras \( A \) and \( B \) are called isomorphic if there is an isomorphism from \( A \) onto \( B \). The name automorphism of \( A \) is reserved for isomorphisms from the algebra \( A \) onto itself.

\textit{\textbf{\( \ast \)-algebras}}

The first appearance of an algebra in quantum theory coincides with the beginning of the modern quantum theory. Heisenberg, in proposing his famous canonical commutation relation \( pq - qp = i/\hbar \), did not think of the observables \( p, q \) to be some infinite matrices or operators in Hilbert space. The possibility to represent \( p \) and \( q \) in such a way was realised a little bit later. (Following M. Born, Heisenberg, at the moment of his great discovery, was not aware of the mathematical existence of such objects.) In this spirit we may think of an algebra as generated by the observables of a certain physical system and later on represent this algebra as an algebra of operators. At the time of the foundation of quantum physics it was not known how to connect the algebraic technique with the powerful methods of analysis.
and toatology. Therefore, the influence of this line of thinking was not so strong. Even to-day it seems that there are more question open than solved!

After these aside-remarks we shall introduce a further algebraic structure. The very origin of it may be the fact that we measure real quantities and thus in our mathematical formalism we need some device to distinguish real from complex numbers, real from complex functions, hermitian matrices and operators from arbitrary ones. This requirement is met by involutions.

Let \( \mathcal{A} \) be an algebra. An involution in \( \mathcal{A} \) is an anti-linear map from \( \mathcal{A} \) onto \( \mathcal{A} \)

\[
(2.1) \quad a \in \mathcal{A} \rightarrow a^* \in \mathcal{A}
\]

satisfying

\[
(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^* \quad \text{ (antilinearity)}
\]

\[
(2.2) \quad (a^*)^* = a
\]

\[
(\alpha b)^* = b^* a^*
\]

Now a \( ^* \)-algebra is an algebra, in which an involution is distinguished.

It is a matter of routine to define a \( ^* \)-homomorphism as an homomorphism of one \( ^* \)-algebra \( \mathcal{A} \) into another one satisfying \( \tau(a^*) = \tau(a)^* \). Similarly we speak of \( ^* \)-isomorphisms, \( ^* \)-automorphisms, \( ^* \)-subalgebras,...

If \( N \) is a subset of a \( ^* \)-algebra, we write

\[
(2.3) \quad N^* = \{ a \in \mathcal{A} : a^* \in N \}
\]

\( N \) is called symmetric in the case \( N = N^* \). As an example,
the kernel of every *-homomorphism is a symmetric ideal. On the other hand, the factor algebra $\mathcal{A}/\mathcal{J}$ is again in a natural way a *-algebra for every symmetric ideal $\mathcal{J}$ of $\mathcal{A}$.

An element $a$ of $\mathcal{A}$ is said to be hermitian, if $a = a^*$. If we are able to interpret $\mathcal{A}$ as an algebra of observables, then we should consider its hermitian elements as observables or at least require hermiticity as a necessary condition for observables. The hermitian elements form a real linear space.

An element $a$ of a *-algebra is called positive, if it can be written as a finite sum of the form

\[
\sum a_j^* a_j, \quad a_j \in \mathcal{A}
\]

One writes $b_i \succ b_2$ iff $b_i - b_2$ is a positive element. It is an important case, when $b_i \succ b_2$ together with $b_2 \succ b_i$ is only possible if $b_i = b_2$. Then the real linear space of hermitian elements becomes a semi-ordered one and the positive elements form a proper cone. In the case we are allowed to interpret $a$ as an observable, $a \succ 0$ means the non-negativity of all measured values in whatever state the physical system is.

3. Some examples

Example 3.1.: A commutative algebra.

Let $\mathcal{T}$ be the phase space of a classical system or, more generally, any local compact topological space. The points of $\mathcal{T}$ correspond to the pure states of the system and an observable attaches to every such state a definite value, the value of this observable in the given state. Demanding that
the observed values are not too different, if the considered states are near neighbours, one may restrict oneself to observables that are continuous functions. The set $C(T)$ of all continuous functions on $T$ is of course an algebra with an involution given by the transition to the complex-conjugate function. For non-compact $T$ the algebra $C(T)$ contains unbounded functions, growing arbitrary strongly. Restrictions on the growth of the functions will lead to subalgebras more appropriate for the purposes of physics.

An important example of such an subalgebra is the algebra $C_c(T)$ that consists of all such functions $f$ of $C(T)$, which approach zero if their argument approaches the "boundary" of $T$.

Example 3.2.: The Borel algebra.

Our next example, the Borel algebra $\beta(X)$, yields, as above, also a commutative $\ast$-algebra and is connected with the theory of Bose-Fock-spaces and Gaussian measures. Let $X$ be a real (!) Hilbert space (or pre-Hilbert-space) with scalar product $\langle x, y \rangle$. Let us first consider an arbitrary bounded function $x \mapsto f(x)$ defined on $X$ and let us define the supremum norm

$$\| f \|_X = \sup_{x \in X} |f(x)| \tag{3.1}$$

Next we denote by $\beta(X)$ the set of all functions on $X$ of the form

$$x \mapsto \sum \alpha_j e^{i \langle x, y_j \rangle}, \text{ finite sum, } y_j \in X \tag{3.2}$$

Under the usual addition and multiplication and complex-conjugation $\beta(X)$ obviously is a $\ast$-algebra.
Then $\beta(X)$ exactly consists of those functions $f$ on $X$, which can be approximated by functions of type (3.2) in the norm (3.1). This means $f \in \beta(X)$ iff for every $\epsilon > 0$ there is one $g \in \beta_\epsilon(X)$ with $\|f - g\|_X < \epsilon$. The construction above is easily generalized to arbitrary real vector spaces. Under the influence of Nelson, Glimm, Jeffe and many others algebras of this type have become an important tool in constructive and euclidean quantum field theory.

Remark: In the algebras $C(T), C_\sigma(T), \beta(X)$ an element of one of these algebras is hermitian, iff it is, considered as a function on $T$ or $X$, a real function. If it is positive in the sense of (2.4), this function will in addition be a non-negative one.

Example 3.3.: The algebra of matrices.

Let us consider the set $B_n$ of all non-matrices. We can add and multiply matrices and the transition to the hermitian conjugate of a matrix defines an involution. Therefore it is natural to consider $B_n$ as a $^*$-algebra. Its hermitian elements coincide with the hermitian matrices. An element of the $^*$-algebra $B_n$ is positive iff the matrix is a positive semi-definite one, i.e. if the eigenvalues are non-negative.

Of course we can think of $B_n$ as of the set of all linear operators of a complex linear Hilbert space of finite dimension $n$. Note that for finite spin-lattice systems or for the spin variables of some particle the interpretation of algebras of the type $B_n$ as algebras of observables is straightforward.
Example 3.4: \( B^* \)- and \( C^* \)-algebras.

Most problems in quantum physics need a complex Hilbert space \( \mathcal{H} \), which is not finite dimensional. Let us denote by \((\xi, \eta)\) its scalar product and write as usual \( \| \xi \| = \sqrt{(\xi, \xi)} \). If \( \alpha \) is a linear operator from \( \mathcal{H} \) onto \( \mathcal{H} \), we define its norm by

\[
\| \alpha \| = \sup \{ \| \alpha \xi \| : \| \xi \| = 1, \xi \in \mathcal{H} \}
\]

and denote by \( \mathcal{B}(\mathcal{H}) \) the set of all bounded operators, i.e., operators with \( \| \alpha \| < \infty \). As in example 3.3 \( \mathcal{B}(\mathcal{H}) \) turns out to be a \(*\)-algebra, the involution of which is defined by

\[
(\xi, \alpha \eta) = (\alpha^* \xi, \eta).
\]

The norm (3.2) has a very important property which is also shared by norms of type (3.1):

\[
\| \alpha^* \alpha \| = \| \alpha \|^2.
\]

Norms with this property are called \( C^* \)-norms.

A subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{H}) \) is said to be a \( B^* \)-algebra or a "concrete \( C^* \)-algebra", if it is closed under (3.2): If \( \alpha_j \in \mathcal{A} \) and \( \alpha \in \mathcal{B}(\mathcal{H}) \) with \( \| \alpha - \alpha_j \| \to 0 \), then \( \alpha \in \mathcal{A} \).

A \( * \)-algebra, \( * \)-isomorphic to a \( B^* \)-algebra, is called a \( C^* \)-algebra. There is a vast literature on the theory of such algebras to which we cannot give proper attention. We only mention one basic result, that historically was the starting point of the theory of such algebras: Gelfand and Neimark discovered that the existence of a \( C^* \)-norm and the closedness of the algebra with respect to this norm is sufficient for a \( * \)-algebra to be \( * \)-isomorphic to a \( B^* \)-algebra and hence to be a \( C^* \)-algebra.
Example 3.5.: \textbf{op}*-algebras.

Most of the physical observables and quantities like differential and field operators cannot be represented with the help of bounded operators. They are essentially unbounded. One way out of this is to look for functions of them that are bounded, in order to fall back into the class of C*-algebras, which behaves mathematically very well. The prize we have then to pay are considerable calculation complications. Therefore it seems to be a good advice, to look for more general algebras of operators. Let us shortly describe, how to do this.

We suppose \( \mathcal{D} \) to be a dense linear submanifold of \( \mathcal{H} \).

An operator is in \( \mathcal{L}_+(\mathcal{D}) \) provided the following is true:

(i) the domain of definition of \( \alpha \) coincides with \( \mathcal{D} \),

(ii) \( \alpha_+ \) maps \( \mathcal{D} \) into \( \mathcal{D} \), \( \alpha \mathcal{D} \subseteq \mathcal{D} \),

(iii) \( \alpha^* \) exists and its domain of definition contains \( \mathcal{D} \).

(iv) \( \alpha^* \) maps \( \mathcal{D} \) into \( \mathcal{D} \), i.e., \( \alpha^* \mathcal{D} \subseteq \mathcal{D} \).

In virtue of these conditions \( \mathcal{L}_+(\mathcal{D}) \) becomes an algebra of operators. By definition denotes \( \alpha^+ \) the restriction of \( \alpha^* \) to \( \mathcal{D} \). It results an involution \( \alpha \mapsto \alpha^+ \) in \( \mathcal{L}_+(\mathcal{D}) \) and the algebra \( \mathcal{L}_+(\mathcal{D}) \) becomes a \( * \)-algebra.

Some properties of \( \mathcal{L}_+(\mathcal{D}) \) are similar to that of \( \mathcal{B}(\mathcal{H}) \).

One can prove, for example, that every \( * \)-automorphism \( \tau \) of \( \mathcal{L}_+(\mathcal{D}) \) is inner, i.e., there is a unitary operator \( u \in \mathcal{L}_+(\mathcal{D}) \) with \( \tau(\alpha) = u^* \alpha u \). If \( \mathcal{D} = \mathcal{H} \) is a Hilbert space, one has \( \mathcal{L}_+(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \).

Finally we define:

An \textbf{op*-algebra} is a \( * \)-subalgebra of an algebra \( \mathcal{L}_+(\mathcal{D}) \) which contains the identity element of \( \mathcal{L}_+(\mathcal{D}) \).
Let us remark that, unlike our procedure in defining $B^\dagger$-algebras, we do not require a closedness condition for $\text{op}^\dagger$-algebras.

Example 3.6.: Algebra of field operators

Algebras of field operators are, if properly defined, a la Wightman, a special sort of $\text{op}^\dagger$-algebras.

Given the components $\bar{\Phi}_1(x), ..., \bar{\Phi}_m(x)$ of some relativistic quantum field, their action as operators in its Hilbert space $\mathcal{H}$ is supposed to be as follows: There is a dense linear submanifold $\mathcal{F}$ of $\mathcal{H}$, a space of test functions $f(x_1, ..., x_n)$ of Minkowski space-time points $x_1, ..., x_n$. (Let us here use for definiteness test-functions of the tempered distributions of Schwartz) and a rule, according to which we have to define field operators $\bar{\Phi}_{i_1...i_n}(f) \in \mathcal{L}_{\mathcal{F}}(\mathcal{F})$.

One assumes that

\begin{enumerate}
  \item[(j)] $f \rightarrow \bar{\Phi}_{i_1...i_n}(f)$ is complex linear in $f$
  \item[(jj)] $f \rightarrow \langle f, \bar{\Phi}_{i_1...i_n}(f) \rangle$ defines a tempered distribution for every vector $\xi \in \mathcal{F}$
  \item[(jjj)] there is a $\mathcal{S}\mathcal{E}\mathcal{S}$-matrix $\mathcal{S}_{ij}$ satisfying
    \[ \sum_k \mathcal{S}_{ik} \mathcal{S}_{kj} = \delta_{ij} \]

such that we always have

\[ \bar{\Phi}^*_{i_1...i_n}(f) = \sum_j \mathcal{S}_{ij} \bar{\Phi}_{i_1...i_n}(f) \]

with

\[ \mathcal{S}(x_1, ..., x_n) = \mathcal{F}(x_1, x_2, ..., x_n) \]

(The bar denotes the complex conjugate of a function.) The set of all such operators, which may also be symbolically written

\[ \bar{\Phi}_{i_1...i_n}(f) = \int \bar{\Phi}_{i_1}(x_1) ... \bar{\Phi}_{i_n}(x_n) f(x_1, ..., x_n) d^nx_1 ... d^nx_n, \]

\[ 1 \leq i_k \leq s, \]

where
generates together with the identity of $\mathcal{L}_s(J)$ an op\textsuperscript{**}-algebra $\mathcal{A}[\Phi_1, \ldots, \Phi_5]$, the algebra of field operators of the fields $\Phi_1, \ldots, \Phi_5$.

3.7. Test-algebra for quantum fields

Our last example is called test-algebra for quantum fields and it is connected with the reconstruction theorem of Wightman: There exists a *-algebra $\mathcal{R}$ such that for every field with components $\Phi_{a_1}, \ldots, \Phi_{a_s}$ satisfying (j), (jj), and (jjj) from the above example one can find a *-homomorphism $\tau$ from $\mathcal{R}$ onto $\mathcal{A}[\Phi_1, \ldots, \Phi_5]$. The elements $f$ of $\mathcal{R}$ are defined to be the formal finite sums

$$f = \lambda e + \sum n \sum i_1 \cdots i_n f_{i_1 \cdots i_n},$$

(finite sum)

where $e$ will be the identity of $\mathcal{R}$, $\lambda$ a complex number and $f_{i_1 \cdots i_n} = f_{i_1 \cdots i_n}(x_1, \ldots, x_n)$ with $1 \leq i_k \leq 5$ are test-functions depending on the space-time points $x_1, \ldots, x_n$.

In (3.5) $f_{i_1 \cdots i_n}$ is called the $(i_1, \ldots, i_n)^{th}$ component of $f$, $\lambda e$ is called its $0^{th}$ component, which is also denoted by $f_{i_1 \cdots i_n}$. $\mathcal{R}$ is in a natural way a complex linear space in which we add two elements by adding componentwise their components. We introduce the involution by

$$f^* = \lambda e + \sum n \sum i_1 \cdots i_n f_{i_1 \cdots i_n}^*,$$

(3.6)

with the help of the matrices $\delta_{i_k}$ occurring in (jjj) of example 3.6. Finally, we explain the multiplication in $\mathcal{R}$:

$$(g \otimes f)_{i_1 \cdots i_n}(x_1, \ldots, x_n) = \sum_{k=0}^{n} g_{i_1 \cdots i_k}(x_1, \ldots, x_n) f_{i_{k+1} \cdots i_n}(x_{k+1}, \ldots, x_n).$$
which determines the \((i_1, \ldots, i_n)^{th}\) component of \(q \circ f\), \(q, f \in R\). Some work with pencil and paper will convince one, that \(R\) is indeed a \(*\)-algebra with respect to the above definitions.

Let us now consider the algebra of field operators \(A[\Phi \bar{\Phi}]\) and define for every \(f \in R\)

\[
\Phi (f) = \lambda 1 + \sum \Phi_{i_1, \ldots, i_n}(t_{i_1} \cdots t_{i_n})
\]

where the symbols in the right-hand side are given in example 3.6 and 1 abbreviates the identity of \(L_1(\mathcal{H})\). Then the following statement is true:

\[
f \mapsto \Phi (f), \quad f \in R
\]

is a \(*\)-homomorphism from \(R\) onto \(A[\Phi, \bar{\Phi}]\).

3.7.: Further examples...

In our sketchy introduction we have not mentioned or have not given due attention to other important classes of \(*\)-algebras: algebras connected with the canonical commutation and anticommutation rules, algebras of differential operators, group algebras, \(*\)-algebras and so on... This richness should not bother one, as it only indicates the importance of the \(*\)-algebra structures which may be well compared with group structures.

4. States

We have already seen how to associate heuristically to certain \(*\)-algebras concepts like "observable", "field operator". They have now to be complemented by the concept of "state" and "expectation value". We need the observables to distinguish
and to characterise states. One may say that in a sense, which becomes clear later on, observables and states as well as field operators and expectation values are dual objects to one another. Given an observable $a$ and a state $\varphi$ of a physical system, we try to find the "expectation value $\varphi(a)$ of $a$ if the system is in the state $\varphi$", that is, the arithmetical mean of the observed values of $a$ in the state $\varphi$.

\[(4.1) \quad a \rightarrow \varphi(a) \leftarrow \varphi\]

We supplement this by the assumption, that there are "enough" observables, so that any two states $\varphi_1, \varphi_2$ can be separated $\varphi_1(a) \neq \varphi_2(a)$ with an appropriate chosen observable $a$.

Being this true, we are allowed to identify mathematically states and their expectation values, i.e., we are allowed to consider a state as a certain functional

\[(4.2) \quad \varphi : \quad a \rightarrow \varphi(a)\]

of all the observables.

What kind of functional (4.2) will be identified with a state? Well, the concept of state (like that of observable) is so rich in structure, that we should not dare to describe this concept in some completeness. One can only try to give some general "necessary condition" for the functional (4.2) and this condition creates a "mathematical concept of state". The mentioned necessary requirements are roughly described as follows. Firstly, (4.2) should be linear in $a$, which means that for any two observables $a$ and $b$ there should
exist one observable called \( a + b \) satisfying \( g(a+b) = g(a) + g(b) \)
for all states. Secondly, the expectation value of an observable should be real-valued. Thirdly, \( g(\alpha) \geq 0 \) if the measured values of \( \alpha \) are supposed to be non-negative in all states, i.e., if we consider the observable to be a "positive" one. Fourthly, we need a normalisation: For any two states \( g_1 \) and \( g_2 \) it should never occur that
\[
g_1(\alpha) = \lambda g_2(\alpha)
\]
for all \( \alpha \) and fixed \( \lambda \).

These requirements are easily fulfilled with the help of a \(*\)-algebra. (In passing we point to more general ways to do this. One can start with a general real semiordered linear space to characterise observables or one starts with a more general convex set to describe states.)

Let \( \mathcal{A} \) be a \(*\)-algebra. A linear functional \( g \) over \( \mathcal{A} \) is a map \( \alpha \mapsto g(\alpha) \) of \( \mathcal{A} \) into the complex numbers with \( g(\alpha + \beta) = g(\alpha) + g(\beta) \), \( g(\lambda \alpha) = \lambda g(\alpha) \). The functional is called hermitian, iff \( g(\alpha^*) = g(\alpha) \) is fulfilled for all \( \alpha \in \mathcal{A} \). It is called positive, iff we have \( g(\alpha) \geq 0 \) for all positive elements \( \alpha \) of \( \mathcal{A} \).

The positive functionals are essentially those we are looking for with the exception of normalisation. Let us now assume that there is an identity element \( e \) in \( \mathcal{A} \). Then we define: A state of \( \mathcal{A} \) is a positive linear functional \( g \) satisfying \( g(e) = 1 \).

If \( g \) is a positive linear functional, then \( \langle a, b \rangle = g(a^* b) \) gives us a positive semidefinite scalar
product over $\mathcal{A}$. Hence the inequality (Bunjakovski, Cauchy, Schwartz, etc. inequality)
\begin{equation}
|\varrho(a^* b)|^2 \leq \varrho(a^* a) \cdot \varrho(b^* b)
\end{equation}
is valid. Here we mention only one application. If $\varrho$ is a state, then (4.3) gives with $b = e$ the inequality
\[ \varrho(a^* a) \geq |\varrho(a)|^2 . \]
Therefore we may define the "uncertainty" of $a$ in the state $\varrho$ by
\begin{equation}
\Delta(\varrho, a) = \sqrt{\varrho(a^* a) - |\varrho(a)|^2}
\end{equation}
which in some cases is also called "fluctuation", "dispersion" or "root-mean-square deviation".... For two hermitian elements $a, b$ of $\mathcal{A}$ a small calculation using (4.3) yields the "uncertainty relation"
\begin{equation}
\Delta(\varrho, a) \cdot \Delta(\varrho, b) \geq \frac{1}{\hbar} |\varrho(a b - b a)|^2
\end{equation}
which is in the case of canonical commutation relation a version of the Heisenberg uncertainty relation.

The set of all states of a $\mathcal{A}$-algebra may be called its state space. Given some states $\varrho_1, \varrho_2, \ldots, \varrho_m$ and some positive numbers $\varrho_1, \varrho_2, \ldots, \varrho_m$ satisfying $\sum \varrho_j = 1$, we get a new state by setting $\varrho = \sum \varrho_j \varrho_j$. $\varrho$ is called a mixture (a la Gibbs) of the states $\varrho_i$ with weights $\varrho_j$. This simple fact is also expressed by saying "the state space is a convex set" and by calling $\varrho$ a "convex linear combination of the $\varrho_i$".

A state is called pure or extremal if every mixture $\varrho = \sum \varrho_j \varrho_j$ with positive weights is trivial, i.e., implies $\varrho_i = \varrho$ for all $i$.

Let us now have a look at our examples. If $\varrho$ is a positive linear functional over $C_c(T)$, then there exists
by the Riesz representation theorem a measure \( \mu \) over \( \mathcal{T} \) with
\[
\mathcal{G}(a) = \int_{\mathcal{T}} a(x) d\mu
\]
Though for non-compact \( \mathcal{T} \) we have no identity in \( C_0(\mathcal{T}) \),
one can normalise \( \mathcal{G} \) demanding \( \mathcal{T} \) to have unity volume
with respect to the measure \( \mu \). Hence \( \mathcal{G} \) is called a state
iff
\[
1 = \int_{\mathcal{T}} d\mu
\]
This fits very well in our picture: Interpreting \( \mathcal{T} \) to be
the phase space of a classical system, the states of \( C_0(\mathcal{T}) \)
are in one-to-one correspondence to the probability distributions over \( \mathcal{T} \), i.e., to the mixed states of the system.
Furthermore, the pure states are given by Dirac measures on
\( \mathcal{T} \): For every pure state \( \sigma \) there is one point \( x \) of
the phase space \( \mathcal{T} \) with \( \sigma(a) = a(x) \) and vice versa.
The algebra \( C(\mathcal{T}) \) admits not so much states: They are also
given by (4.6) but with the subsidiary condition that \( \mu \)
is concentrated on a compact subset of \( \mathcal{T} \), because an
arbitrary finite measure on \( \mathcal{T} \) may have infinite expecta-
tion values for some unbounded functions.
As a rule, properly chosen commutative \( \ast \)-algebras are good
for describing classical systems, especially in statistical
mechanics.
Example 3.2 reduces in principle to that of 3.1 by an important
theorem due to Gelfand, which provides us with a \( \ast \)-isomorphism
onto a certain algebra \( C(\mathcal{T}) \). However, the space \( \mathcal{T} \)
occurring here, is terribly complicated and cannot be given in explicitly enough terms. Let us mention only one class of states: If \( \langle i, \eta \rangle \) is a positive definite (or semidefinite) quadratic form defined on the real Hilbert space \( \mathcal{H} \), then there exists a state \( \varphi \) over \( \mathcal{A}(\mathcal{H}) \) which fulfills
\[
\varphi(\alpha) = \exp \{-\frac{1}{2} \langle \eta, \eta \rangle \}
\]
with \( \alpha(i) = \exp \{i \langle i, \eta \rangle \} \) and which is called a "Gaussian functional of \( \mathcal{A}(\mathcal{H}) \)".

Let us now consider the examples 3.3 and 3.4, starting with \( \mathcal{A}(\mathcal{H}) \). Let \( d \) be a density operator of \( \mathcal{H} \), i.e. a positive hermitian operator with trace one. Then
\[
(4.9) \quad \varphi : \quad \varphi(\alpha) = \text{tr} \alpha d
\]
is a state over \( \mathcal{A}(\mathcal{H}) \). Hence some states correspond uniquely to the density operators and these states are called normal ones. If \( \mathcal{H} \) is finite dimensional, every state is normal, but otherwise there exists in addition a large set of the so-called singular states. The later have expectation value zero for every finite dimensional projection operator. A general state is a mixture of a normal and a singular one.

Further, if \( \xi \in \mathcal{H} \) is a normalised vector, \( (\xi, \xi) = 1 \), then
\[
(4.10) \quad \sigma : \quad \sigma(\alpha) = (\xi, \alpha \xi)
\]
is a pure state. Every normal pure state of \( \mathcal{B}(\mathcal{H}) \) is of the form (4.10).

Let us now shortly consider a \( \mathcal{B}^* \)-algebra \( \mathcal{A} \) which is in \( \mathcal{B}(\mathcal{H}) \) and contains the identity of \( \mathcal{B}(\mathcal{H}) \). If then \( \varphi \) is a state of \( \mathcal{B}(\mathcal{H}) \), its restriction \( \varphi_0 \) onto the elements of \( \mathcal{A} \) clearly yealds a state of \( \mathcal{A} \). One can show that
every state of $\mathcal{A}$ can be obtained by this procedure.

There is not known a general way to obtain states of an arbitrary $\ast$-algebra. For op-$\ast$-algebras one always can find a lot of states having a representation like (4.9). Though the problem of finding and describing states is much more difficult in the unbounded case, it is encouraging to find also some simplifications. Indeed, for some choices of the domain $\mathcal{D}$ in the definition of $\mathcal{L}_+(\mathcal{D})$ in example 3.5 the $\ast$-algebra $\mathcal{L}_+(\mathcal{D})$ has only states which are given by density operators and its pure states are completely given by (4.10) with $\xi \in \mathcal{D}$. To construct such a domain $\mathcal{D}$ we take a self-adjoint operator $\alpha$ with a discrete spectrum $\lambda_1, \lambda_2, \ldots$ satisfying $\sum \lambda_j^{-m} < \infty$ for a natural number $m$. Then $\mathcal{D} = \cap \mathcal{D}_n$, $\mathcal{D}_n$ being the domain of definition of $\alpha^n$.

What about the test algebra $\mathcal{R}$? If $\tau$ is a $\ast$-homomorphism of $\mathcal{R}$ into $\mathcal{A}$ and if $\xi_0$ is a state of $\mathcal{A}$, then $\xi(\alpha) = \xi_0(\tau(\alpha))$, $\alpha \in \mathcal{R}$ defines a state of $\mathcal{R}$. Due to its "universal character", $\mathcal{R}$ should allow extremely many states, though the known results in this direction are not so easy to obtain and yet not sufficiently strong for the purposes of physics.

5. Representations

Up to now we have introduced states and their mixing and we are going to give attention to the superposition of states. To do this, we consider representations of $\ast$-algebras,
i.e., we look for possibilities to realise a \( \ast \)-algebra as an algebra of operators.

A \( \ast \)-representation of a \( \ast \)-algebra \( \mathcal{A} \) is a \( \ast \)-homomorphism \( \tau \) into an algebra \( \mathcal{L}_+(\mathcal{H}) \). \( \mathcal{H} \) sometimes is called domain of definition of \( \tau \) and the Hilbert space \( \mathcal{H} \), in which \( \mathcal{H} \) is a dense manifold, is said to be the representation space of \( \tau \). For every \( \xi \in \mathcal{H} \), \( (\xi,\xi) = 1 \) the functional

\[
\mathcal{g}_\xi(\alpha) = (\xi,\alpha^* \xi), \quad \alpha \in \mathcal{A}
\]

is a state of \( \mathcal{A} \) and this is described by saying that \( \mathcal{g}_\xi \) is a vector state of the representation \( \tau \).

The question arises whether we can represent every state as a vector state. Yes, we can. One can even construct a \( \ast \)-representation \( \hat{\tau} \) of \( \mathcal{A} \) in such a way, that every state of \( \mathcal{A} \) is represented as a vector state of \( \hat{\tau} \). These representations are, however, often too highly "discontinuous".

We are now able to define transition probabilities. Given two states \( \mathcal{g}_1 \) and \( \mathcal{g}_2 \) of \( \mathcal{A} \), we define \( P_{\mathcal{g}_1,\mathcal{g}_2} \), the transition probability from \( \mathcal{g}_1 \) to \( \mathcal{g}_2 \) to be the supremum of all numbers \( |(\xi,\eta)|^2 \), where \( \xi, \eta \) run over all such pairs of vectors of all \( \ast \)-representations \( \tau \) of \( \mathcal{A} \), for which \( \mathcal{g}_1(\alpha) = (\xi,\tau(\alpha) \xi) \) and \( \mathcal{g}_2(\alpha) = (\eta,\tau(\alpha) \eta) \).

Clearly,

\[
P_{\mathcal{g}_1,\mathcal{g}_2} = P_{\mathcal{g}_2,\mathcal{g}_1}, \quad 0 \leq P_{\mathcal{g}_1,\mathcal{g}_2} \leq 1, \quad P_{\mathcal{g},\mathcal{g}} = 1.
\]

Examples:

1) The transition probability of two different pure states of a commutative \( \ast \)-algebra is always zero. (There is no superposition in classical mechanics!) 2) Take \( \xi, \eta \in \mathcal{H} \) and \( \mathcal{A} = \mathcal{L}_+(\mathcal{H}) \).
and denote by $\varphi_1, \varphi_2$ the associated vector states \((5.1)\) of $\mathcal{L}_+(\mathcal{H})$. Then one has $P_{\varphi_1, \varphi_2}\mathcal{P}_{(\ell, \eta)} = |(\ell, \eta)|^2$ if $(\ell, \eta) = (\eta, \eta) \neq \eta$.

3) Let $\varphi_1, \varphi_2$ be two normal states of $\mathcal{B}(\mathcal{H})$ with density operators $d_1, d_2$. It follows $P_{\varphi_1, \varphi_2} = tr(d_1 d_2^*) d_2^*$.

We finish our short excursus to $\ast$-representations with the famous GNS-construction (after Gelfand, Naimark, Segal). Let $\tau$ be a $\ast$-homomorphism of $\mathcal{A}$ into $\mathcal{L}_+(\mathcal{H})$ and let us choose a vector $\xi_0 \in \mathcal{H}$. Define $\mathfrak{J}_0 = \{ \xi_0 \}$ to be the set of vectors $\xi_0$, which can be realized as $\xi_0 = \tau(\alpha) \xi_0$, $\alpha \in \mathcal{A}$. Suppose that $\mathcal{A}$ has an identity, so that $\xi_0 \in \mathcal{H}$. If $\mathfrak{J}_0 = \mathcal{H}$, we call $\xi_0$ cyclic vector of $\tau$.

Let us now assume $\xi_0$ to be cyclic, $(\xi_0, \xi_0) = 1$ and denote by $\varphi_0$ the vector functional associated with $\xi_0$ by \((5.1)\).

Suppose now $\tau'$ to be another $\ast$-representation of $\mathcal{A}$ into an algebra $\mathcal{L}_+(\mathcal{H}')$. Suppose further, that the vector $\xi'_0$ is cyclic for $\tau'$ and that the vector functional associated with $\xi'_0$ equals $\varphi_0$. Then $\tau$ and $\tau'$ are equivalent representations in the following sense. Let us choose $\xi \in \mathfrak{J}$.

There is $\alpha \in \mathcal{A}$ with $\xi = \tau(\alpha) \xi_0$. Define $u \xi = \xi'$ with $\xi' = \tau'(\alpha) \xi_0$. Then $u$ is well defined, maps $\mathfrak{J}$ isometrically onto $\mathfrak{J}'$ and fulfills $u \tau(\alpha) = \tau'(\alpha) u$. (u induces a $\ast$-isomorphism from $\mathcal{L}_+(\mathcal{H})$ onto $\mathcal{L}_+(\mathcal{H}')$ with $\tau' = \tau \cdot u$.)

The considerations above show that a state $\varphi_0$ uniquely defines, up to equivalence, a $\ast$-representation $\tau$ with the cyclic vector $\xi_0$ and $\varphi_0(\alpha) = (\xi_0, \tau(\alpha) \xi_0)$. The usual way to construct this representation is as follows. Define

\[(5.3) \quad \mathfrak{J} = \{ \xi \in \mathcal{A} : \varphi_0(\alpha \xi) = 0 \text{ all } \alpha \in \mathcal{A} \}\]
$J$ is a subalgebra of $A$ with the property $a b \in J$ if only $b \in J$ and $a \in A$. Because of this property, $J$ is called a left ideal. Defining $\xi_a = a + J$, so that $\xi_a = \xi_a'$ is the same as $a' - a \in J$, we get

\begin{equation}
\xi_a + \xi_{a'} = \xi_{a' + a'}, \quad \lambda \xi_a = \xi_{\lambda a}.
\end{equation}

The set of all "classes modulo $\mathcal{R} = \xi_a"$ is, according to (5.4), a complex linear vector space $\mathcal{F}$. The scalar product

\begin{equation}
\langle \xi_a, \xi_{a'} \rangle = \theta(a^*_a a_{a'}),
\end{equation}

is uniquely defined and positive definite on $\mathcal{F}$. The GNS-representation $\tau$ given by $\theta_0$ now reads

\begin{equation}
\alpha \rightarrow \tau(a) \quad \text{with} \quad \tau(a) \xi_a = \xi_{aa'},
\end{equation}

and one easily checks that (5.6) is a *-representation of $A$ into $L_+ (\mathcal{D})$. The cyclic vector is given by $\xi_e$, $e$ being the identity of $A$. All vector states of the GNS-representation given by $\theta_0$ are obtained by the ansatz

\begin{equation}
\theta (a^*_a a_{a'}) = \theta_0 (b^*_a b_{a'}) = \theta_0 (b_{a'} b_a) \quad \text{with} \quad b \in A, b \notin J.
\end{equation}

Let us now consider an example. In the Hilbert space $\mathcal{H}$ of one-particle amplitudes $f(k)$ with scalar product

\begin{equation}
(f, g) = \int \bar{f}(k) g(k) \frac{d^3 k}{\sqrt{k^2 + m^2}},
\end{equation}

we have a real subspace $\mathcal{X}$ consisting of all real-valued functions $f(k)$. Consider the Bohr algebra $\beta(\mathcal{X})$ and the state $\theta_0$ defined as in (4.6) by

\begin{equation}
\theta_0 (e^{i\varphi}) = \exp \left\{ -\frac{i}{2} (\varphi, \varphi) \right\}, \quad \varphi = \theta (f) = (f, g)
\end{equation}

with $\varphi \in \mathcal{X}$. Carrying out the GNS-construction, denoting by
\[ \Omega \text{ its cyclic vector and writing} \]

(5.9) \[ \tau(e^{i\phi}) = e^{i \hat{\Phi}(\xi)} \]

we have

(5.10) \[ \langle \Omega, e^{i \hat{\Phi}(\xi)} \Omega \rangle = \exp \left\{ -\frac{i}{2} \langle \xi, \xi \rangle \right\} \]

This relation characterises the Bose-Fock space with vacuum \( \Omega \), one-particle space \( \mathcal{H} \), and free field operator

(5.11) \[ \hat{\Phi}(\xi) = \int \hat{\phi}(\xi') \phi(\xi') d^3 \xi' \]

Hence the result of the GNS-construction with the Gaussian functional \( \mathfrak{g} \) over \( \mathcal{B}(\mathcal{H}) \) yields the Bose-Fock space in the so-called "Q-space" representation. Constructions of such a kind have been used by Segal, Glimm, Jaffe, Nelson and others in its approach to "constructive quantum field theory". One has to find states over \( \mathcal{B}(\mathcal{H}) \), different from the Gaussian ones and satisfying some rather natural conditions to obtain by the GNS-construction of such a functional quantum fields "with interaction". To-day, thanks to the initiating work of the authors mentioned above, this program can be carried through at least for some models in two space-time dimensions.

\( \mathfrak{g}_1 \)-Continuity

By the very nature of physical reasoning and to come into contact with analytic methods one has to perform limiting procedures and to make continuity assumptions and considerations. In some *-algebras, for example in C*-algebras and the algebras \( \mathcal{L}_+(\mathcal{H}) \), it is intrinsically prescribed already by
their algebraic structure, how one has to handle correctly continuity arguments. In other ones, however, there is some or even large freedom, as in the case of test algebras. The general methods can be found in every book on topology. As to the way of presentation, we follow an idea of Treves.

In a given complex linear space $\mathcal{M}$ we consider seminorms, which are real-valued functions $\xi \mapsto q(\xi)$, $\xi \in \mathcal{M}$ with

$$q(\xi) \geq 0, \quad q(\lambda \xi) = |\lambda| q(\xi), \quad q(\xi + \gamma) \leq q(\xi) + q(\gamma).$$

A set $\mathcal{F}$ of seminorms is called a topology (a "local convex" one, to be more precise) if the following is true:

(i) If $q \in \mathcal{F}$ and the seminorm $q_*$ satisfies $q_*(\xi) \leq q(\xi)$ for all $\xi \in \mathcal{M}$, then $q_* \in \mathcal{M}$.

(ii) From $q_1, q_2 \in \mathcal{F}$ it follows $q_1 + q_2 \in \mathcal{F}$.

(iii) For every $\xi \neq 0$ there is at least one $q \in \mathcal{F}$ with $q(\xi) 
eq 0$.

A topology $\mathcal{F}$ is generated by a set of seminorms $\{q_\alpha\}$, if $\mathcal{F}$ is the smallest set of seminorms containing all $q_\alpha$ and fulfilling (i) and (ii). We have to satisfy (iii) already by the set $\{q_\alpha\}$. If $\mathcal{F}$ can be generated by one single seminorm $q$, then $\mathcal{F}$ is called a norm and $\mathcal{F}$ is a norm topology. In a $C^*$-algebra we have a $C^*$-norm topology...

A subset $\mathcal{U} \subseteq \mathcal{M}$ is called closed with respect to $\mathcal{F}$ ("$\mathcal{F}$-closed") iff from $\xi_0 \not\in \mathcal{U}$ there follows the existence of a $q \in \mathcal{F}$ with $q(\xi) > q(\xi_0)$ for all $\xi$ obeying $q(\xi - \xi_0) < 1$.

$\mathcal{U}$ is called dense in $\mathcal{M}$, if there is no other closed subset of $\mathcal{M}$ then $\mathcal{M}$ itself, in which $\mathcal{U}$ is contained.
We now consider two linear spaces $M_1$ and $M_2$ with topologies $T_1$ and $T_2$. A map $\tau$ from $M_1$ into $M_2$ is called continuous, if for every $q_2 \in T_2$ the seminorm $q_2(\tau(f))$ is in $T_1$. [Because of property (i) above it is enough to show $\lambda q_1(f) \geq q_2(\tau(f))$ for a seminorm $q_1$ of $T_1$. Hence we can check continuity with the help of a generating system of seminorms of $T_1$ and of $T_2$.]

The last point of the general introduction is a rather special concept. $T$ is called barreled, if every seminorm $q_0$, for which the set of all $f$ satisfying $q_0(f) \leq 1$ is $T$-closed, is always contained in $T$.

A topological $\ast$-algebra is a $\ast$-algebra $A$ together with a topology $T$ such, that with $q \in T$ also the seminorms

\[(6.2) \quad \alpha \mapsto q(\alpha^*), \alpha \mapsto q(b\alpha), \alpha \mapsto q(\alpha b)\]

are contained in $T$ for every $b \in A$. The continuity of the maps $\alpha \mapsto \alpha^*, \alpha \mapsto b\alpha, \alpha \mapsto \alpha b$ is then guaranteed.

All this applies to every op-$\ast$-algebra, which is by definition contained in an algebra $L_+(D)$. To introduce the lessner topology or uniform topology $T_4$ of $A$ we first define

\[(6.3) \quad q_n(\alpha) = \sup_{\xi \in \pi} |\langle \xi, \alpha \eta \rangle|\]

for every subset $\pi$ of $T$. $T_4$ is generated by all such $q_n$, which turn out to be seminorms, i.e., for which for all $\alpha \in A$ we have $q_n(\alpha) < \infty$.

These topologies work nicely because of the following facts:
one can recover the Hausdorff topology of \( L^*_1(\mathcal{A}) \) purely algebraically, not knowing anything about \( \mathcal{F} \) in advance.

2) Every closed *-subalgebra \( \mathcal{A} \) of \( L^*_1(\mathcal{A}) \), for which \( \mathcal{F} \) is a Hilbert space, will be equipped with its usual C*-norm topology by \( \mathcal{F}' \), i.e., \( \mathcal{F}' \) is generated by the C*-norm.

3) Assuming \( \mathcal{A} \subseteq \mathcal{A}' \), we get in an obvious sense \( \mathcal{F}' \subseteq \mathcal{F}' \): In going to smaller algebras the topology becomes "stronger".

4) In the situation just described let \( \mathcal{A} \) be barrelled with respect to \( \mathcal{F}' \). Then \( \mathcal{F}' = \mathcal{F}' \) if the seminorms of \( \mathcal{F}' \) are considered only for elements of \( \mathcal{A} \).

5) The cone of positive elements of \( \mathcal{A} \) is normal with respect to \( \mathcal{F}' \), i.e., there is a generating set of seminorms \( \{ q_x \} \) with \( q_x(a + b) \geq q_x(a) \) for all positive elements \( a, b \) of \( \mathcal{A} \).

What is an important concerning 4) and 5) is the following result of Schmüdgen (which covers also the C*-case):

Let \( \mathcal{A} \) be a *-algebra with identity element and with a topological structure \( \mathcal{F} \). If then \( \mathcal{A} \) is a barrelled topological *-algebra and if its cone of positive elements is a normal one, then there exists a bicontinuous *-isomorphism onto an on *-algebra \( \mathcal{A} \) with topological structure \( \mathcal{F} \).

1.3. Local structures, linear conditions

As a rule, not every state of a *-algebra (of observables, field operators or test functions) has some physical meaning, is a "physical" state. The reason is that in the definition of states such important requirements as symmetry principles,
positiveness of energy and mass spectra, causality conditions, continuity and analyticity requirements and similar things are not included. Therefore we have to select out of the state space of the \( \mathcal{A} \)-algebra in question the states satisfying a given set of subsidiary conditions. Generally speaking, these conditions are "linear"ones: It is possible to define a linear subspace \( \mathcal{M} \) of hermitian linear functionals of the \( \mathcal{A} \)-algebra with the property that a general state fulfills the desired requirements, if and only if the state is contained in \( \mathcal{M} \). Because of this, one has only one non-linear condition to satisfy, the positivity condition. In virtue of this, one can sometimes divide a complicated investigation into two parts: the "linear program" or the construction of \( \mathcal{M} \) as explicit as possible and the "non-linear program" or the search for those solutions of the linear program which are states. In the Wightman frame, for instance, the linear program consists in finding all Lorentz-invariant distributions which fulfill the locality (Einstein causality) and spectrality axioms and in exploring their properties. This is already an extremely complicated task, despite its mentioned "linear character"!

Let us make one further concrete step to meet physical demands. Let \( \mathcal{A} \) be a \( \mathcal{A} \)-algebra and \( T \) be the Euclidean or Minkowskian space (or a lattice or even a general topological space). A local structure in \( \mathcal{A} \) over \( T \) is given by associating to every open set \( \sigma \) of \( T \) a \( \mathcal{A} \)-subalgebra
\( \mathcal{A}(\mathcal{O}) \) of \( \mathcal{A} \) with the monotony property

\[(7.1) \quad \mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) \subset \mathcal{A}(\mathcal{O}_1) \cup \mathcal{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2 \]

We have further to add a continuity requirement. Assuming that \( \mathcal{A} \) is a topological *-algebra and assuming that the union of the open sets \( \mathcal{O}_1, \mathcal{O}_2, \ldots \) of \( \mathcal{T} \) is equal to \( \mathcal{O} \), the algebra generated by the subalgebras \( \mathcal{A}(\mathcal{O}_j), \ j = 1, 2, \ldots \) should be dense in \( \mathcal{A}(\mathcal{O}) \). If \( \mathcal{N} \) is any subset of \( \mathcal{T} \) we may define \( \mathcal{A}(\mathcal{N}) \) to be the intersection of all algebras \( \mathcal{A}(\mathcal{O}) \) with \( \mathcal{N} \subset \mathcal{O} \), \( \mathcal{O} \) open.

In statistical physics, where \( \mathcal{T} \) is an Euclidean space or a lattice, one asks for a local structure with \( \alpha_1 \alpha_2 = \alpha_2 \alpha_1 \) for all \( \alpha_1, \in \mathcal{A}(\mathcal{O}_1), \alpha_2 \in \mathcal{A}(\mathcal{O}_2) \) if \( \mathcal{O}_1 \cap \mathcal{O}_2 \) is empty.

It is similar with the algebra of field operators \( \mathcal{A}[\mathbb{F}_1, \ldots, \mathbb{F}_n] \) which we inspect a bit closer. In this case \( \mathcal{T} = \mathcal{M} \) is the Minkowskian space. The support \( \text{supp} \ f \) of a test function \( f : (\mathbb{R}_1, \ldots, \mathbb{R}_n) \) of \( n \) space-time points is the smallest closed subset \( \mathcal{N} \) of \( \mathcal{M} \) with the property: From \( f : (\mathbb{R}_1, \ldots, \mathbb{R}_n) \neq 0 \) it follows \( x_j \in \mathcal{N}, \ j = 1, 2, \ldots, n \). Now \( \mathcal{A}(\mathcal{O}) \) is defined to be the subalgebra of \( \mathcal{A}[\mathbb{F}_1, \ldots, \mathbb{F}_n] \) generated by the identity and all field operators \( \mathbb{F}_i \cdot \mathbb{F}_j \) for which \( \text{supp} \ f \subset \mathcal{O} \). In the Bose case, to which we restrict ourselves, the fields are as usual called local fields, if \( \alpha_1 \alpha_2 = \alpha_2 \alpha_1 \) for \( \alpha_1, \alpha_2 \in \mathcal{A}(\mathcal{O}_j) \) whenever the space-time regions \( \mathcal{O}_1, \mathcal{O}_2 \) are space-like separated. The same procedure holds for the test algebra \( \mathcal{R} \). For every \( f \in \mathcal{R} \) the smallest closed set of space-time points containing the supports of all components of
\( \mathcal{F} \) is called support of \( f \) and is also written \( \text{supp } f \).

The subalgebra, generated by all \( f \) with \( \text{supp } f \subseteq \mathcal{O} \), is open, is the building stone \( R(\mathcal{O}) \) of the local structure of \( R \).

Let us now consider the \( \ast \)-homomorphism \( \tau \) (eq.(3.9)) from \( R \) onto an algebra of Bose field operators. Clearly, \( \tau \) maps \( R(\mathcal{O}) \) onto \( A(\mathcal{O}) \) and preserves the local structures.

Let us consider the kernel \( J \) of the homomorphism \( \tau \). If the supports of \( f \) and \( f' \) are space-like separated, then the commutator of the field operators \( \Phi(f) \) and \( \Phi(f') \) have to vanish. Hence \( f \otimes f' - f' \otimes f \in J \). Because \( J \) is an ideal, we consider the ideal \( J_0 \), generated by all such commutators \( f \otimes f' - f' \otimes f \) with space-like separated supports. It is \( J_0 \subseteq J \) and \( J_0 \) is called locality ideal of \( R \). One necessary condition for a state \( \xi \) of \( R \) to become a state of an algebra of local Bose fields is therefore \( \xi(f) = 0 \) for all \( f \in J_0 \). If we could use \( R/J_0 \) instead of \( R \) in the construction of fields, we satisfied automatically the locality axiom. However, the structure of \( R/J_0 \) is not known.

We now combine local structures with symmetries. Let \( A \) be a \( \ast \)-algebra with a local structure over \( T \). A symmetry, compatible with this local structure (sometimes also called "covariance system") consists of i) a group \( \Gamma \) of \( \ast \)-automorphism of \( A \) and ii) a group homomorphism \( \sigma \rightarrow \bar{\sigma} \) from \( \Gamma \) into a transformation group \( \bar{\Gamma} \) of the space \( T \) such that every \( \sigma \in \Gamma \) maps \( A(\mathcal{O}) \) onto \( A(\bar{\sigma} \mathcal{O}) \), where \( \bar{\sigma} \mathcal{O} \) is the set of all space-time points \( \bar{\sigma} x \) with \( x \in \mathcal{O} \).
A good example is the test algebra $R$ to see this mechanism. Let $\Gamma$ be the Poincaré group and let us define the action of $\sigma$ on an element $\eta$ of $R$ componentwise by

\[(7.2) \left(\sigma^{-1}\eta_{k_{1}-i_{1}}\right)(x_{1}, \ldots, x_{n}) = \sum_{k} \eta_{k_{1}}(\sigma) \cdots \eta_{k_{n}}(\sigma) \eta_{k_{1}-i_{1}}(\sigma x_{1}, \ldots, \sigma x_{n})\]

Then the support of $\sigma \eta$ is the transform of supp $\eta$ under $\sigma$ and this shows the compatibility of the symmetry with the local structure. (The matrices $H_{ij}$ form a representation of the Lorentz group of finite dimension. In order to define $\ast$-automorphisms by (7.2), they have to obey the relation $\sum_{c} \lambda_{c} \text{ Tr}_{c} \Phi_{c} = \sum_{c} \lambda_{c} \text{ Tr}_{c} \Phi_{c}$.)

Let us now consider a state $\Omega$ over $R$ with zero expectation values for the locality ideal and assume the invariance of $\Omega$ under $\Gamma$:

\[(7.3) \quad \Omega(\eta) = \Omega(\sigma \eta) \quad \text{all} \quad \sigma \in \Gamma\]

In performing the GNS-construction with $\Omega$, domain $D$ and cyclic vector $\Omega_{0}$ we arrive at a certain algebra $A$ of local field operators with the local structure described above. Now $\Omega$ is invariant and the uniqueness, up to equivalence, of the GNS-construction comes into action: It guarantees the existence of a unitary representation of the Poincaré group $\sigma \rightarrow \mathcal{U}(\sigma), \sigma \in \Gamma$ which leaves $D$ invariant and fulfills the compatibility condition for the new algebra $A$ because of

\[(7.4) \quad \mathcal{U}(\sigma) \Phi(\eta) \mathcal{U}(\sigma)^{-1} = \Phi(\sigma \eta), \mathcal{U}(\sigma) \Omega_{0} = \Omega_{0}\]
Having reached this state of affairs, only the spectrality properties remain open from the fundamental requirements for a relativistic local Bose quantum field. These, however, are in the first instance requirements to the unitary representation \( \hat{\sigma} \rightarrow U(\hat{\sigma}) \), restricted to the translational subgroup of \( \mathbb{R} \). If \( \xi \) is a four vector and \( U(\xi) \) the representative of the translation by \( \xi \), we have to demand
\[
\int g(\xi) U(\xi) d^4\xi = 0
\]
for every absolutely integrable function \( g \), the Fourier transform \( \hat{g} \) of which is vanishing on the closed forward cone of momentum space. This condition can be pulled back to the test algebra \( \mathcal{R} \). Applying (7.4) one sees the equivalence of (7.5) with
\[
(7.6) \quad \tilde{\Phi}(\xi') \delta_{\omega} = 0, \quad \mathcal{R}_{i_{n-1} \cdots i_1}(\lambda_1 \cdots \lambda_n) = \int g(\xi) \mathcal{R}_{i_{n-1} \cdots i_1}(\xi') d^4\xi
\]
This can be expressed in terms of \( g \). Let us call \( \mathcal{J}_o \) the set of all such elements of \( \mathcal{R} \) for which every of their components \( \mathcal{R}_{i_{n-1} \cdots i_1}(\lambda_1 \cdots \lambda_n) \) fulfill the condition: If for the Fourier transform \( \tilde{\mathcal{R}}_{i_{n-1} \cdots i_1}(\xi') \) of \( \mathcal{R}_{i_{n-1} \cdots i_1}(\xi') \) we have \( \tilde{\mathcal{R}}_{i_{n-1} \cdots i_1}(\lambda_1 \cdots \lambda_n) \neq 0 \), then at least one of the four vectors \( k_{n-1}, k_n \cdots k_{n-1}, k_{n}, k_{n+1}, \ldots k_{n-1} \) is not in the closed forward cone. The zero component of \( \xi \) as to be zero. This set \( \mathcal{J}_o \) is a left ideal and our \( g \) gives a theory with spectrality condition by the GNS-construction, if \( g \) vanishes for all elements of \( \mathcal{J}_o \).

The fact, that \( \mathcal{J}_o \) is a left ideal, has an interesting consequence: Denote by \( g = \sum p_j \xi_j \) a mixture of states
with positive weights. The vanishing of \( \Omega \) for the elements of \( J_0 \) implies the same for all the states \( \varphi_i \).

Now we may summarise the statements about \( R \): There is a linear space of functionals \( M \) such that \( \Omega \) is a \text{Tightman functional} if it is a state and if it is in \( M \).

\( M \) is of the form \( M = M_a \land M_b \land M_c \). \( M_a \) contains the invariant functionals, satisfying (7.3). \( M_b \) and \( M_c \) contain the functionals, vanishing on \( J_0 \) and \( J_0 \) respectively. This illustrates what was said at the beginning of this section.

\section*{References}

We only recommend some articles with review character and text books, which provides a more complete and more specialised introduction into topics touched here.


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