ATTEMPT TO CONSTRUCT TEST-ALGEBRAS FOR
HAAG-ARAKI FIELDS

Armin Uhlmann

Karl-Marx-University, Leipzig G.D.R.

1. Basic notations

We have to introduce the "algebra of test-functions" \([1,2]\),
called \( \mathcal{R} \) and an extension \( \overline{\mathcal{R}} \) of \( \mathcal{R} \). Though the algebra \( \overline{\mathcal{R}} \) is
not of direct physical importance, \( \overline{\mathcal{R}} \) contains some interesting
subalgebras.

Let us denote with \( S_n \) the linear space equipped with the
Schwartz topology of test-functions for the tempered distributions of \( n \) space-time points. \( S_0 \) denotes the field of complex
numbers. Now we consider "functions" \( \tilde{a} \) over the non-negative
integers, the values of which for the integer \( n \) is in \( S_n \):

\[
(1.1) \quad a: \ n \rightarrow \tilde{a}(n) = a(n; x_1, x_2, \ldots, x_n) \in S_n
\]

We denote the set of all such functions by \( \overline{\mathcal{R}} \). Obviously \( \overline{\mathcal{R}} \)
becomes a linear manifold if we consider the mappings \((1.1)\) to
be linear maps from \( \overline{\mathcal{R}} \) onto \( S_n \). Indeed, \( \overline{\mathcal{R}} \) is just the di-
rect product.
\( (1.2) \quad \tilde{R} = \prod S_n, \quad n = 0, 1, 2, \ldots \)

and we introduce the direct product topology in \( \tilde{R} \). Thus \( \tilde{R} \) is a complete local convex linear space. Now for \( a, b \in \tilde{R} \) we define the product \( a \cdot b \) to be the map

\( (1.3) \quad ab: \quad n \to \sum a(r) \times b(s); \quad r + s = n \)

with

\( (1.4) \quad a(r) \times b(s) = a(r; x_1, \ldots, x_r) b(s; x_{r+1}, \ldots, x_{r+s}) \)

By this multiplication \( \tilde{R} \) becomes an algebra with unit element \( e \) that is defined by \( e(\emptyset) = 1, e(n) = 0 \) for \( n \neq \emptyset \).

The imbedding given by (1.4) of \( S_n \times S_m \) into \( S_{n+m} \) is continuous and hence the multiplication in the algebra \( \tilde{R} \) is continuous simultaneously in both factors. Finally \( \tilde{R} \) becomes symmetric \([3]\) by the definition

\( (1.5) \quad \tilde{a}^* : n \to \tilde{a}(n; x_1, x_{n-1}, \ldots, x_0) \).

The bar denotes the complex conjugate of the function \( a(n) \).

We mention two simple properties of \( \tilde{R} \):

a/ There are no zero-divisors in \( \tilde{R} \).

Proof: If the two elements \( a, b \) are not identically zero, we consider the smallest numbers \( r \) and \( s \) with \( a(r) \neq 0 \) and \( b(s) \neq 0 \).

Then \( (ab)(r+s) = a(r) \otimes b(s) \) is not the zero of \( S_{r+s} \).
Hence $ab \neq 0$.

(b) If $a \notin \bar{R}$ and $a(s) \neq 0$ then $a^{-1} \in \bar{R}$.

Proof: We may assume $a(s) = 1$. Then the sequence

$$(1.6) \quad b_n = \frac{1}{s} (e-a)^n, \quad n = 1, 2, ...$$

converges in $\bar{R}$ because $b_n(s) = b_n(\infty)$ for all $s \geq n$.

Remark $(e-a)^j(s) = 0$ for $j > n$. Therefore the limit $b$ of (1.6) is determined by $b(s) = b_n(s)$. Now we have

$$b_n(e-a) = (e-a) b_n = b_{n+1} - e$$

In going to the limit we obtain $ba = ab = e$.

Remark 1: If $a^{-1}$ exists in $\bar{R}$ for an element $a \in \bar{R}$,
then there is a neighbourhood $V_{\lambda}$ of $a$ such that the map $b \rightarrow b^{-1}$ with $b \in V_{\lambda}$ exists and is bicontinuous.

Remark 2: If $a \in \bar{R}$ and $a(s) = 0$ then for every formal power series $p(x)$ the series $p(a)$ converges in $\bar{R}$.

The support $\text{supp}_a$ of an element $a \in \bar{R}$ is defined to be the smallest closed subset of the Minkowski space with the property

$$(1.7) \quad \text{supp}_a(x_1, x_2, ..., x_n) \subseteq \text{supp}_a \otimes \text{supp}_a$$

with the $n$-fold product on the right hand side for $n \geq 1$.

Let be $\mathcal{O}$ an open set of the Minkowski space. The set $\bar{R}(\mathcal{O})$ of all $a \in \bar{R}$ with $\text{supp}_a \subseteq \mathcal{O}$ is a symmetric subalgebra of $\bar{R}$. Given an arbitrary subset $\Delta$ of the Minkowski space,
we define

\[(1.8) \quad \mathcal{R}(\Delta) = \bigcap \mathcal{R}(\mathcal{O}) ; \Delta \subseteq \mathcal{O} , \quad \mathcal{O} \text{ open}.\]

A continuous linear map \( \tau \) from \( \mathcal{R} \) into \( \mathcal{R} \)

\[\tau : a \rightarrow a^\tau\]

is called endomorphism of \( \mathcal{R} \), if

\[(ab)^\tau = a^\tau b^\tau .\]

If \( \tau \) is not identically zero we have \( e^\tau = e \). Indeed let be \( a^\tau \neq 0 \). It follows \( a^\tau = e^\tau a^\tau \) and \( (e-e^\tau) a^\tau = 0 \).

Now there are no zero divisors in \( \mathcal{R} \). Hence \( e = e^\tau \).

If we always have

\[(a^\tau)^* = (a^*)^\tau\]

the endomorphism is called symmetric. An endomorphism is called "automorphism" if it maps \( \mathcal{R} \) onto \( \mathcal{R} \). An important example of a group of symmetric automorphisms is the set of automorphisms induced on \( \mathcal{R} \) by the transformations of the inhomogeneous Lorentz group. The connected component of the identity of this automorphism group will be denoted by \( \Gamma \) and its translation subgroup by \( \Gamma^c \).

Now we define \( \mathcal{R} \), the algebra of test functions. Algebraically \( \mathcal{R} \) is a symmetric subalgebra of \( \mathcal{R} \). An element \( a \in \mathcal{R} \) is in \( \mathcal{R} \) if there is an integer \( n_0 \) with \( a(n) = 0 \) for all \( n > n_0 \). The smallest integer \( n_0 \) with this property
is the "degree of a". The zero of $\mathcal{R}$ is said to have the
degree $-\infty$. In $\mathcal{R}$ one has to introduce a stronger topology
than the one induced in $\mathcal{R}$ as a subset of $\bar{\mathcal{R}}$. Namely we
consider $\mathcal{R}$ as the direct sum

$$ (1.9) \quad \mathcal{R} = \mathbb{Z} \oplus S_n, $$

equipped with the direct sum topology. In this topology $\mathcal{R}$ is
complete and the product (1.3) turns out to be continuous in
both factors simultaneously. With respect to a point set $\Delta$
of the Minkowski space we introduce the symmetric subalgebras

$$ (1.10) \quad \mathcal{R}(\Delta) = \mathcal{R} \cap \bar{\mathcal{R}}(\Delta) $$

Let us further notice, that the restrictions on $\mathcal{R}$ of the
automorphisms $\Gamma$ may be considered as continuous automor-
phisms of $\mathcal{R}$ and we shall use the same notation $\Gamma$ for
them.

Now we denote by $\mathcal{K}_0$ the set of all elements of $\mathcal{R}$ which
can be represented in the form of a finite sum

$$ (1.11) \quad \sum a_i^* a_i \quad \text{with} \quad a_i \in \mathcal{R} $$

A linear form $A$ of $\mathcal{R}$ is "positive", i.e. fulfills Wightman's
positivity condition, iff

$$ (1.12) \quad \langle A, a \rangle \geq \sigma \quad \forall a \in \mathcal{K}_0 $$
The closure $\mathcal{K}$ of $\mathcal{K}$ with respect to the direct sum topology also enjoys the property

$$(1.13) \quad \langle A, a \rangle \geq 0 \quad \forall a \in \mathcal{K}$$

with respect to every continuous positive linear form $A$. For every continuous symmetric endomorphisms $\tau$ of $\mathcal{R}$ we have $\mathcal{K}^\tau \subseteq \mathcal{K}$. Further it can easily be seen that the sum of two elements of $\mathcal{K}$ is in $\mathcal{K}$ and that with $\lambda > 0$ also $\lambda a$ is in $\mathcal{K}$ for non-negative real $\lambda$. The more the intersection of $\mathcal{K}$ with $-\mathcal{K}$ consists only of the zero of $\mathcal{R}$. For a proof see [4].

2. The subalgebras $\mathcal{R}(\Lambda)$ of $\mathcal{R}$

Using the imbedding in $\mathcal{R}$ of $\mathcal{R}$ one can construct in various ways symmetric algebras, which may be of interest in quantum field theory. Indeed, to formulate the usual axioms [5] of quantum field theory it is sufficient to have a symmetric/topological/ algebra together with:

1/ a realisation of the Poincaré group in terms of /continuous/ symmetric automorphisms of the algebra in question and

2/ a notation of "support in space-time" for the elements of the algebra, that is compatible with the automorphisms mentioned in 1/. Obviously, such an algebra will serve as an "algebra of test-functions".
In the following we consider special examples of such algebras. Let us call admissible a subset $N$ of $R$ with the following properties:

i/ $e \in N$

ii/ $NN \subseteq N$, i.e. $N$ is multiplicatively closed

iii/ $N^\omega = N$ i.e. with $a \in N$ also $a^\omega \in N$.

iv/ $a(e) = e$ for all $a \in N$

v/ $a^\tau \in N$ for all $a \in N$ and all $\tau \in \Gamma$.

To every admissible subset $N$ of $R$ one can associate a symmetric subalgebra $R(N)$ of $R$ in a natural way: An element $b$ of $\overline{R}$ is contained in $R(N)$ if and only if one can write it in the form of a finite sum

$$ (2.1) \quad b = \sum \lambda_i a_i^{-1} \quad \text{with} \quad a_i \in N $$

with complex numbers $\lambda_i$. To comment on this we note first that by virtue of property iv/ every $a \in N$ has an inverse. Because of ii/ the product of two elements of the form (2.1) is again in $R(N)$ and for the sum of two elements (2.1) and for the multiples with complex numbers this assertion is trivial. Hence $R(N)$ is an algebra. This algebra is symmetric because of iii/.

iv/ tells us that there is a unit element in $R(N)$ and finally it follows from v/ that $R(N)$ admits $\Gamma$ as an automorphism group. For an arbitrary point set $\Delta$ of the Minkowski space we define

$$ (2.2) \quad R(N, \Delta) = R(N) \cap \overline{R}(\Delta) $$
3. Pre-norms

Let \( N \) be an admissible subset of \( R \). A real valued function \( \gamma \) on \( N \)

\[(3.1) \quad \gamma : a \rightarrow \gamma(a), \ a \in N\]

is called pre-norm, if

\[(3.2) \quad \gamma(a) \geq 0 \quad \text{for all } a \in N\]

\[(3.3) \quad \gamma(e) = 1\]

\[(3.4) \quad \gamma(a) = \gamma(a^*)\]

\[(3.5) \quad \gamma(ab) \leq \gamma(a)\gamma(b)\]

Given two pre-norms \( \gamma_1 \) and \( \gamma_2 \) the pre-norm \( \gamma_2 \) is called "stronger" /more exactly "not weaker"/ than \( \gamma_1 \) if \( \gamma_1(a) \geq \gamma_2(a) \) for all \( a \in N \). Consider a pre-norm \( \gamma \) and define

\[(3.6) \quad \tilde{\gamma}(a) = \inf \{ \gamma(a_1) \ldots \gamma(a_s) \} \]

with \( a_i \in N \) and \( a = a_1a_2\ldots a_s \)

The function \( a \rightarrow \tilde{\gamma}(a) \) is a pre-norm again and \( \tilde{\gamma} \) is called "the regularisation of the pre-norm \( \gamma \)". A regular pre-norm
is a pre-norm, which equals its regularisation. Note that
the regularisation \( \overline{g} \) of a pre-norm \( g \) is the strongest
regular pre-norm, which is weaker than the original pre-norm
\( g \). An element \( a \in N \) is called \( N \)-prim, if there does not
exists a decomposition \( a = a_1 a_2 \) with \( a_i \in N \) and
\( a_i \neq e \) for \( i = 1, 2 \). If \( a \) is \( N \)-prim, so does \( a^* \).
Define

\[
(3.7) \quad N_\mathfrak{p} = \{ a \in N ; a \text{ is } \mathfrak{p} \text{-prim} \}
\]

and consider on \( N_\mathfrak{p} \) a real valued function \( a \rightarrow g_\mathfrak{p}(a) \)
satisfying \( g_\mathfrak{p}(a) = g_\mathfrak{p}(a^*) > 0 \). Then, setting \( g_\mathfrak{p}(e) = 1 \),
we construct

\[
(3.8) \quad g(a) = \inf \{ g_\mathfrak{p}(a_1) \ldots g_\mathfrak{p}(a_k) \}
\]

with \( a_i \in N_\mathfrak{p} \cup \{ e \} \) and \( a = a_1 a_2 \ldots a_k \).

If it turns out that if \( g(a) \neq 0 \) on \( N_\mathfrak{p} \), the function \( g \)
is a regular pre-norm and every regular pre-norm may be ob-
tained in this way. Of course \( g(a) = 0 \) implies for \( a \) the
existence of an infinite number of different decompositions
\( a = a_1 \ldots a_k \) with \( N \)-prim elements \( a_k \). It is most likely
that such a situation can not occur at all. However, we are
able to prove this only for a restricted class of admissib-
le subsets \( N \) of \( \mathbb{R} \). A pre-norm \( g \) is called
\( \Gamma \)-invariant, if

\[
(3.9) \quad g(a) = g(a^\tau) \quad \text{for all } \tau \in \Gamma
\]
If \( \mathcal{g} \) is \( \Gamma \)-invariant, then the same is true with its
regularisation \( \mathcal{g}^* \). To construct \( \Gamma \)-invariant regular
pre-norms one has simply to use a \( \Gamma \)-invariant function
\( \mathcal{g} \) on \( \mathbb{N}_p \) in the formula (3.8).

Remark: There is an obvious gap in the definition of pre-
norms: it is desirable to have a smoothness condition for \( \mathcal{g} \). Probably the following definition
is a relevant one. We call \( \mathcal{g} \) smooth, if the set

\[
N(\mathcal{g}, \lambda) = \{ a \in \mathbb{N} : \mathcal{g}(a) \leq \lambda \}
\]

is relatively closed for every real positive \( \lambda \)
i.e. there exist in \( \mathbb{R} \) subsets \( M_\lambda \) which are closed in the direct sum topology and satisfying

\[
N(\mathcal{g}, \lambda) = N \cap M_\lambda
\]

(3.11)

4. The Banach algebras \( \mathcal{Q}(\mathbb{N}, \mathcal{g}) \).

Let \( \mathcal{g} \) be a pre-norm on \( \mathbb{N} \). We denote by

\[
\mathcal{Q}(\mathbb{N}, \mathcal{g})
\]

the set of all complex-valued functions defined on \( \mathbb{N} \):

\[
(f : a \rightarrow f(a) , \ a \in \mathbb{N})
\]

(4.1)

enjoying the property
This implies the vanishing of \( f \) for all \( a \in \mathbb{N} \) with the exception of a countable set. Now we consider \( g \) as a measure on \( \mathbb{N} \) that gives the value
\[
\sum a(a) = a \in \mathbb{N}
\]
for the set \( a \subseteq \mathbb{N} / \mathbb{N} \) is equipped with the discrete topology. Clearly, \( Q(N, g) \) consists just of the absolute integrable functions integrable with respect to the "measure" \( g \). Therefore with the norm

\[
(4.3) \quad g(f) = \sum_{a \in \mathbb{N}} |f(a)| g(a)
\]

\( Q(N, g) \) becomes a Banach space.

Now we define an involution \( f \rightarrow f^* \) by

\[
(4.4) \quad f^* : a \rightarrow f(a^*)
\]

The involution maps \( Q(N, g) \) onto itself and preserves the norm:

\[
(4.5) \quad g(f) = g(f^*)
\]

Next we introduce a multiplication between the elements of \( Q(N, g) \). With every pair of elements \( f, g \) of \( Q(N, g) \) we associate the function \( fg \) defined as following

\[
(4.6) \quad (fg)(a) = \sum f(a_i) g(a_2) \quad \text{with} \quad a_i \in \mathbb{N} \quad \text{and} \quad a = a_2 a_1
\]
The following estimate proves that the product \((4.0)\) is well defined and belongs to \(Q(N, \mathfrak{g})\) again:

\[
g(\mathfrak{f} \mathfrak{g}) = \sum_a | \sum_{a_1, a_2} f(a_1) g(a_2) | g(a_2 a_1)
\]

\[
\leq \sum_{a_1, a_2} | f(a_1) g(a_2) | g(a_2 a_1)
\]

\[
\leq \sum_{a_1, a_2} | f(a_1) | g(a_1) | \cdot | g(a_2) | g(a_2)
\]

\[
= g(\mathfrak{f}) g(\mathfrak{g})
\]

Thus we have seen the absolute convergence of \((4.0)\) and the relation

\[
(4.7) \quad g(\mathfrak{f} \mathfrak{g}) \leq g(\mathfrak{f}) g(\mathfrak{g})
\]

In \(Q(N, \mathfrak{g})\) there is a unit element. This we denote by \(1\) and it is given by

\[
(4.8) \quad 1(e) = 1 \quad \text{and} \quad 1(a) = 0 \quad \text{for} \quad a \neq e
\]

Hence we have

Lemma 1. \(Q(N, \mathfrak{g})\) is a symmetric Banach algebra with unit element.

Now turning to property \(\nu)\) for admissible sets \(N\) we see that there is a natural action of the group \(\Gamma\) on \(Q(N, \mathfrak{g})\) provided \(\mathfrak{g}\) is \(\Gamma\)-invariant. In this case \(\Gamma\) is reali-
ized as a group of symmetric automorphisms of $Q(N, g)$ by defining $f^\tau$ to be the map

\[(4.9) \quad f^\tau : a \rightarrow f(a^\tau), \quad f \in Q(N, g)\]

for $a \in N$ and $\tau \in \Gamma$. Because of (3.9) obviously always

\[(4.10) \quad g(f^\tau) = g(f)\]

Hence $\Gamma$ can be considered as group of isometric automorphisms of $Q(N, g)$ provided $g$ is $\Gamma$-invariant.

Finally we have to introduce the notion of support for the elements of $Q(N, g)$. If $f \in Q(N, g)$ than supp. $f$ is defined to be the closure of the union of all supp. $f$ satisfying $f(a) \neq 0$.

Now supp. $a$ for any $a \in \mathcal{R}$ is the closure of an open set in Minkowski space. Hence supp. $f$ is the union of at most countable many such sets. Therefore

\[(4.11) \quad \text{supp} \, f = \text{closure} \, \text{interior of supp} \, f \, f \, f \, f \, f \]

For an open set $\sigma$ of the Minkowski space we have

\[(4.12) \quad Q(N, g, \sigma) = \{ f \in Q(N, g) : \text{supp} \, f \in \sigma \} \]

and for an arbitrary set $\Delta$ of space-time points

\[(4.13) \quad Q(N, g, \Delta) = \bigcap Q(N, g, \sigma) \]
with $\Delta \subseteq \mathcal{O}$ and $\mathcal{O}$ open.
Now we may state (4.13) in another form. $f \in \mathcal{Q}(N, g, \Delta)$
is equivalent with $\text{supp } f \subseteq \mathcal{O}$ for all $\Delta \subseteq \mathcal{O}$, $\mathcal{O}$ open.
But every set is the intersection of open sets. Hence
\[(4.14) \quad \mathcal{Q}(N, g, \Delta) = \left\{ f \in \mathcal{Q}(N, g) : \text{supp } f \subseteq \Delta \right\}\]
for all sets $\Delta$ of Minkowski space.
In this way we see easily
\[(4.15) \quad \mathcal{Q}(N, g, \Delta_1) \subseteq \mathcal{Q}(N, g, \Delta_2) \quad \text{if} \quad \Delta_1 \subseteq \Delta_2\]
and furthermore
\[(4.16) \quad \mathcal{Q}(N, g, \bigcap \Delta_k) = \bigcap \mathcal{Q}(N, g, \Delta_k)\]
where $\{ \Delta_k \}$ denotes an indexed system of subsets of the
Minkowski space. Let us note two consequences of (4.11).
Firstly
\[(4.17) \quad \mathcal{Q}(N, g, \Delta) \subseteq \mathcal{Q}(N, g, \overset{\sim}{\Delta})\]
with $\overset{\sim}{\Delta} = \text{closure } \{ \text{interior of } \Delta \}$
Proof: If $f \in \mathcal{Q}(N, g, \Delta)$ then the interior of supp. $f$ is
contained in the interior of $\Delta$ and hence supp. $f$ is con-
tained in the interior of $\Delta$ and hence supp. $f$ is con-
tained in the closure of the interior of $\Delta$. From (4.14)
we see especially, that $\mathcal{Q}(N, g, \Delta)$ consists of the elements
\[ \lambda^1 \] only if \( \Delta \) is nowhere dense in Minkowski space.

Secondly let \( \mathcal{O} \) be an open bounded set of space-time points. If we have for the open sets

\[ (4.18) \quad \mathcal{O}_i \subseteq \mathcal{O}_{i+1}, \quad \cup \mathcal{O}_i = \mathcal{O} \]

we conclude

\[ (4.19) \quad \mathcal{Q}(N,g,\mathcal{O}_i) = \mathcal{Q}(N,g,\mathcal{O}) \]

Indeed, if \( f \in \mathcal{Q}(N,g,\mathcal{O}) \) then supp. \( f \) is compact and the system of the sets \( \mathcal{O}_i \) is an open covering of supp. \( f \). Therefore a finite number of the \( \mathcal{O}_i \) is sufficient to cover supp. \( f \). Because of the first condition of (4.18) we have supp. \( f \subseteq \mathcal{O}_{i_*} \) for a certain index \( i_* \). Let us further mention the action of an automorphism \( \tau \in \mathcal{T} \):

\[ (4.20) \quad \tau : \mathcal{Q}(N,g,\Lambda) \rightarrow \mathcal{Q}(N,g,\Lambda^\tau) \]

The corresponding element of the Poincaré group is denoted by \( \tau \) also. One may consider \( \mathcal{Q}(N,g) \) as the closure with respect to \( g \) of a graded algebra. To explain this, we consider an element \( a \in \mathcal{R} \) which is not the zero of \( \mathcal{R} \). Let us define \([a]\) the largest of the natural numbers \( s \) with the property: There is decomposition

\[ a = a_1 a_2 \ldots a_s \quad , \quad a_i \in \mathcal{R} \]

and the number of factors \( a_j \) with \( a_j \neq \lambda_j e \), \( \lambda_j \)
complex numbers equals $s$. For example $[c] = e$, $[a] = 1$
if $a$ is of degree one. We now refer to the following lemma [see [6]] for a proof:

**Lemma 2:** Let be $a, b \in \mathbb{R}$ and $ab \neq 0$ then

$$[ab] = [a] + [b]$$

Now let be $f \in \mathcal{G}(\mathbb{N}, \mathbb{G})$ and $\lambda$ a complex number with $|\lambda| \leq 1$. We define an endomorphism

$$(4.21) \quad f \mapsto \lambda \circ f$$

by

$$(4.22) \quad \lambda \circ f : a \mapsto \lambda [a] f(a) \quad \text{for all } a \in \mathbb{N}$$

With the aid of lemma 2 we see by straightforward calculation that $(4.21)$ defines an endomorphism and that

$$(4.23) \quad g(\lambda \circ f) \leq g(f)$$

$$(4.24) \quad (\lambda \circ f)^* = \overline{\lambda} \circ f^*$$

$$(4.25) \quad (\lambda \circ f)^\tau = \lambda \circ f^\tau \quad \text{for all } \tau \in \Gamma$$

$$(4.26) \quad \text{supp} (\lambda \circ f) = \text{supp} f$$

is valid. Moreover, define for any $f \in \mathcal{G}(\mathbb{N}, \mathbb{G})$ the element $f_a$ to be the map
(4.27) \[ a \rightarrow f(a) \quad \text{if} \quad a \in \mathbb{N}, \quad [a] = k \]
\[ a \rightarrow 0 \quad \text{otherwise} \]

If otherwise

Obviously

(4.28) \[ \sum f_n = f \quad \text{with} \quad \sum g(f_n) = g(f) \]

and

(4.29) \[ \lambda \ast f = \sum \lambda^k f_k \]

Therefore for any continuous linear form \( \varphi \) of \( Q(N, g) \) the expression \( \langle \varphi, \lambda \ast f \rangle \) is holomorphic in \( \lambda \) for \( |\lambda| < 1 \) and

(4.30) \[ \langle \varphi, \lambda \ast f \rangle = \sum \lambda^k \langle \varphi, f_k \rangle \]

If therefore \( \langle \varphi, \lambda \ast f \rangle = 0 \) for real \( \lambda \), \( |\lambda| < 1 \) it follows \( \langle \varphi, f_k \rangle = 0 \), \( k = 0, 1, 2, \ldots \)

Now let be \( \mathcal{V} \) closed subspace of \( Q(N, g) \) with \( \lambda \ast \mathcal{V} \subseteq \mathcal{V} \) for real \( |\lambda| < 1 \). If \( f \in \mathcal{V} \) and \( f_k \) is defined by (4.27), then \( f_k \in \mathcal{V} \). Namely every continuous linear form which is zero on \( \mathcal{V} \) is zero for the elements \( f_k \) by the above arguments. Hence because \( \mathcal{V} \) is closed, \( f_k \in \mathcal{V} \)

Shortly, a closed linear subspace \( \mathcal{V} \) is generated by elements with the property

\[ \lambda \ast f = \lambda^k f \quad , \quad |\lambda| < 1 \]
if and only if \( \lambda \in \mathcal{N} \) \( \in \mathcal{N} \) for real \( \lambda \), \( |\lambda| < 1 \).

We shall call such a subspace an homogeneous one. A function, defined on \( \mathcal{N} \) and being zero with the exception of at most a finite subset of \( \mathcal{N} \) is called finite. The set \( \mathcal{Q}_0(\mathcal{N}) \) of finite functions on \( \mathcal{N} \) is a subset of \( \mathcal{Q}(\mathcal{N}, g) \). The more \( \mathcal{Q}_*(\mathcal{N}) \) is a symmetric subalgebra of \( \mathcal{Q}(\mathcal{N}, g) \) which is dense in \( \mathcal{Q}(\mathcal{N}, g) \).

\[(4.31) \quad \mathcal{Q}(\mathcal{N}, g) = \text{closure in } \mathcal{Q}(\mathcal{N}, g) \text{ of } \mathcal{Q}_0(\mathcal{N})\]

There is a natural homomorphism of \( \mathcal{Q}_*(\mathcal{N}) \) onto \( \mathcal{R}(\mathcal{N}) \) defined by

\[(4.32) \quad \chi_* : f \rightarrow \sum f(a)\bar{a}^{-1}, f \in \mathcal{Q}_*(\mathcal{N})\]

Let us further denote the kernel of this homomorphism by \( I_*(\mathcal{N}) \). Unfortunately only rather trivial things are known about the structure of the ideal \( I_*(\mathcal{N}) \). For example, every \( \tau \in \mathcal{T} \) may be considered as an automorphism of \( \mathcal{Q}_*(\mathcal{N}) \) as well as of \( \mathcal{R}(\mathcal{N}) \). It is for \( f \in \mathcal{Q}_*(\mathcal{N}) \)

\[(4.34) \quad \chi_*(f^\tau) = (\chi_*f)^\tau \text{ for all } \tau \in \mathcal{T}\]

and therefore

\[(4.35) \quad I_*(\mathcal{N})^\tau = I_*(\mathcal{N}) \text{ for all } \tau \in \mathcal{T}\]
From the definition of supp, we see

\[(4.30) \quad \text{supp } f \supset \text{supp } \left( \chi \ast f \right), f \in \mathbb{Q}_e(N)\]

because supp. \(a = \text{supp. } a^{-1} \) for all \(a \in \mathbb{R}, a(\ast) \neq 0\).

Now let be \(f_1, f_2 \in \mathbb{I}_e(N)\) and assume \(\text{supp. } f_1 \cap \text{supp. } f_2\) to be nowhere dense. It follows from \((4.30)\) that \(\text{supp. } (\chi \ast f_1) \cap \text{supp. } (\chi \ast f_2)\) is nowhere dense. But \(\chi \ast f_1 = -\chi \ast f_2\) and thus the supports of \(\chi \ast f_1\) are equal and nowhere dense. Therefore the support of \(\chi \ast f_1\) vanishes and \(\chi \ast f_2\) is a multiple of \(e\). This is equivalent with

Lemma 3. If \(f_1, f_2 \in \mathbb{I}_e(N)\) but \(f_1 - f_1(e) \cdot 1 \notin \mathbb{I}_e(N)\)

for two elements \(f_1, f_2\) of \(\mathbb{Q}_e(N)\) then

\(\text{supp. } f_1 \cap \text{supp. } f_2\)

contains an inner point.

Let us consider a further property of \(\mathbb{I}_e(N)\). The factor algebra \(\mathbb{Q}_e(N)/\mathbb{I}_e(N)\) is isomorphic to \(\mathbb{R}(N)\) and therefore contains no divisors of the zero. The conclusion is: if \(f, g \in \mathbb{I}_e(N)\) then neither \(f\) nor \(g\) is contained in \(\mathbb{I}_e(N)\). In other words, \(\mathbb{I}_e(N)\) is a prime ideal.

Lastly let us mention, that under \((4.32)\) the spectral behaviour of the elements will be changed. Consider an element \(f\) which is different from zero only for one \(a \in N\) with \(f(a) = 1\). We have

\[\chi_e(f \ast \lambda 1) = a^{-1} - \lambda e = \left( (e - \lambda a)^{-1} e \right)^{-1} f^{-1}\]

Therefore, if with \(a \in N\) also \((1 - \lambda)^{-1}(e - \lambda a) \in N\)
the element \( \lambda_0 (f - \lambda 1) \) is invertible in \( \mathbb{Q}/I_0 \).

5. Positive linear forms

The reducing ideal of \( \mathbb{Q}(N, \mathfrak{g}) \) is defined to be the set of all \( f \in \mathbb{Q}(N, \mathfrak{g}) \) with \( \langle \varphi, f \rangle < 0 \) for all positive continuous linear forms of \( \mathbb{Q}(N, \mathfrak{g}) \).

Lemma 4. The reducing ideal of \( \mathbb{Q}(N, \mathfrak{g}) \) is homogeneous. To prove this, we remark that \( f \mapsto \lambda \circ f \) defines for real numbers smaller than a symmetric endomorphism of \( \mathbb{Q}(N, \mathfrak{g}) \). Hence with \( \varphi \) the linear form \( \lambda \circ \varphi \) is a positive one too. Hence the reducing ideal is homogeneous. We now get to show some cases with trivial reducing ideal and list necessary condition for this.

Assumption 1: The pre-norm \( \mathfrak{g} \) is a regular one.

Assumption 2: The elements of \( N_\mathfrak{p} \) are strongly prime [6].

Remark: An element \( q \) of \( \mathcal{R} \) is said to be strongly prime, if firstly \( ab \in \mathcal{R}, a \mathcal{R}, b \mathcal{R} \) implies \( b \in a \mathcal{R} \) and if secondly \( ab \in \mathcal{R}, a \mathcal{R}, b \mathcal{R} \) implies \( b \in a \mathcal{R} \). It can be shown [6], that an element \( q \) of \( \mathcal{R} \) the highest component of which is prime /indecomposable/ is strongly prime. This applies especially to every element of the first degree.

Assumption 3: The elements of \( N_\mathfrak{p} \) are normal ones, i.e.

\[
(5.1) \quad q q^* = q^* q \quad \text{for all } q \in N_\mathfrak{p}.
\]

We denote by \( \mathcal{L}^0(N, \mathfrak{g}) \) the closed ideal of \( \mathbb{Q}(N, \mathfrak{g}) \).
generated by the elements \( \frac{f}{g} \cdot \frac{g}{f} \) with \( \text{supp} f \sim \text{supp} g \).

With \( \mathcal{L}(N, g) \) we denote the closure in \( \mathcal{Q}(N, g) \) of \( \mathcal{L}^0(N, g) \) with respect of the following set of seminorms:

\[
\| f \|_n^2 = \sum_{[a] + n} g(a)^2 | f(a) |^2
\]

Clearly, the system

\[
\Delta \rightarrow \mathcal{Q}(N, g) \cap \mathcal{L}(N, g) \quad \text{def} \quad \mathcal{B}(N, g, \Delta)
\]

where \( \Delta \) runs over the /open/ sets of Minkowski space is a local system of normed symmetric algebras contained in

\[
\mathcal{B}(N, g) = \mathcal{Q}(N, g) / \mathcal{L}(N, g)
\]

Lemma 5. Under the assumptions one to three the algebras \( \mathcal{Q}(N, g) \) and \( \mathcal{B}(N, g) \) are reduced ones, i.e., for every of its elements there exists a continuous linear form, nonvanishing at the given element.

Proof. Because of assumption 2 the elements of \( N \) allow for an almost unique prime factor decomposition /see lemma 6/: two such decompositions differ only by different ordering of the factors, hence

\[
\mathcal{g}(ab) = \mathcal{g}(a) \mathcal{g}(b)
\]

because of assumption 1.
Now consider in \( \mathbb{R} \) the group generated by the elements \( N \).
This group we call \( G \). We construct in the usual manner [3] a faithful representation of \( G \). We denote by \( H \) the Hilbert space consisting of all complex valued functions on \( G \) satisfying

\[
(5.6) \quad \sum_{a \in G} |\xi(a)|^2 < \infty.
\]

The scalar product is of course

\[
(5.9) \quad (\xi, \eta) = \sum \overline{\xi(a)} \eta(a).
\]

For every \( b \) from \( G \) the operator

\[
(5.10) \quad U(b) : \xi(a) \rightarrow \xi(ba)
\]

is unitary. The more,

\[
(5.11) \quad b \rightarrow U(b)
\]

is a faithful representation of \( G \). Now we try the following ansatz for \( f \in \mathcal{Q}(N, g) \):

\[
(5.12) \quad A(f) = f(e) U(e) + \sum_{a + e} \frac{f(a)}{2 \alpha} [U(a^\alpha_t) + U(a)]...
\]

with

\[a_1, a_2 \ldots a_n = a ; \quad a_i \in N_p\]
For a moment let us assume that this is a good definition, then obviously

\[(5.13) \quad \| A(f) \| \leq \sum |f(a)| g(a) = g(f)\]

\[(5.14) \quad A(f^*) = A^*(f)\]

By simple algebraics one also finds

\[(5.15) \quad A(fg) = A(f)A(g)\]

In general, however, \((5.12)\) is ill defined. But the peculiar assumptions two and three for \(N\) prevent us from this desease. Indeed, one can prove \([6]\).

Lemma 6: Let be

\[(5.16) \quad a_1a_2\ldots a_n = b_1b_2\ldots b_n\]

with strongly prime \(a_i\) and \(b_k\). Then the \(b_k\) are a permutation of the \(a_i\). All relations of the form \((5.16)\) are consequences of relations of the form

\[(5.17) \quad a_na_m = a_ma_n; a_n, a_m\text{ strongly prime}\]

Furthermore, if the \(a_j\) are normal elements this implies
(5.19) \[ a_n a_m = a_m a_n \]

In this way we see that (5.12) defines a representation of \( Q(N, \xi) \) in \( H \). Now let be \( \xi \) homogeneous of degree \( n \). Consider

\[ (5.19) \quad A(\xi) \xi = \eta ; \xi_\xi(x) = 1 ; \xi_\xi(a) = 0 \quad \text{otherwise} \]

Let be \( a = a_1 a_2 \ldots a_n \) an element of \( N \) of degree \( n \) with strongly prime \( a_k \) for all \( k \). Then

\[ (5.20) \quad \eta(a) = 2^{-n} s(a) f(a) \]

and we have

\[ (5.21) \quad \| D(f) \| \geq \| D(\xi) \| \geq 2^{-n} \sqrt{\sum_{[a]=n} f(a)^2 s^2(a)} \]

Obviously this is a lower bound for the minimal regular norm in \( Q(N, \xi) \). Now because \( f \) has been chosen homogeneous but otherwise arbitrary, the reducing ideal does not contain homogeneous elements different from the zero. Hence by lemma 4 the assertion is proved for \( Q(N, \xi) \).

Now \( L(N, \xi) \) is homogeneous and because of (5.21) and the closure of this ideal under the seminorms (5.2) the maximal homogeneous subspaces of \( L(N, \xi) \) are complete with respect to the minimal regular norm of \( Q(N, \xi) \). Hence in \( Q(N, \xi) \) the ideal \( L(N, \xi) \) is closed with
respect to the minimal regular norm, and consequently \( Q(N, g)/\mathcal{L}(N, g) \) is reduced.

6. Elements, bounded from below

Here we consider elements of \( R \) which are in a more or less heuristic sense - bounded from below: In every representation \( A \) of \( R \) by unbounded operators in a Hilbert space, such an element \( a \) gives rise to an operator \( A(a) \) with the following property: There exists an bounded operator \( B \) with \( B \cdot A(a) = \text{identity in the domain of definition of } A \). First we define the sets

\[
M_1 = \{ a \in R : a^*a - \lambda e \in K \text{ for certain real } \lambda > 0 \}
\]

\[(6.1)\]

\[
M_2 = \{ a \in R : aa^* - \lambda e \in K \text{ for certain real } \lambda > 0 \}
\]

Obviously

\[
(6.2) \quad M_2 = \{ a \in R : a^* \in M_1, a \in M_1^* = M_1^* \}
\]

Because the zero-components are non-negative for the elements of \( K \), the zero-component of every element of \( M_j \) is non-vanishing. Clearly, \( M_j \) is closed with respect to the multiplication with non-vanishing constants. Furthermore, \( M_j^\tau \subset M_j \) if \( \tau \) is a symmetric endomorphism, that is not identical zero.
Finally, we consider two elements $a_1$ and $a_2$ of $M$. Then

\[(a_1a_2)^*a_1a_2 - \lambda_1\lambda_2 = a_1^*(a_2a_1 - \lambda_1\varepsilon)a_2 + \lambda_1(a_2^*a_2 - \lambda_2\varepsilon)\in K\]

for suitable $\lambda_i > 0$. Therefore $M$ is a multiplicatively closed set /a semigroup with respect to the multiplication of $R$. and because of (3.2) the same is true for $M$. There are, of course, plenty of elements in the sets $M_j$. For instance consider an hermitian element $a = a^*$ and a purely imaginary number $\mu$. Then $a + \mu e$ is in $M_1 \cap M_2$. This is also true for $a \in K$ and real $\mu > 0$. Now we define

\[(6.3) \quad M = \{a \in R : a \in M_1 \cap M_2, a(e) = 1\}.\]

The considerations above show

\[(6.4) \quad M^* = M, \quad M \cdot M \subseteq M\]

i.e. $M$ is closed with respect to the involution and the multiplication of $R$. For every symmetric endomorphism $\tau \neq 0$ we have

\[(6.5) \quad M^\tau \subseteq M.\]

Hence $M$ is an admissable set. Consider a symmetric

/continuous/ representation
(6.8) \[ A : a \rightarrow A(a), \quad a \in \mathbb{R} \]

of \( \mathbb{R} \) into a Hilbert space \( \mathcal{H} \). This implies that

1) there exists a dense linear submanifold \( D \) of \( \mathcal{H} \)

which is the common domain of definition of all the

operators \( A(a) \) and that

2) \( D \) is stable under \( A(a) : A(a) D \subseteq D \)

The meaning of the terminus "symmetric" is \( A(a^*) \in \mathcal{A}^*(a) \)

For every \( a \in \mathbb{R} \) the map

(6.7) \[ a \rightarrow (\omega, A(a) \omega) \]

determines a positive linear form on \( \mathbb{R} \). Let us consider this in the case \( a \in \mathcal{M} \). Then

(6.8) \[ (\omega, A(a^*a - \lambda e) \omega) \geq 0, \quad \lambda > 0 \]

with certain \( \lambda \) and all \( \omega \in D \). Hence

(6.9) \[ ||A(a)\omega|| \geq \lambda \cdot ||\omega||, \quad \forall \omega \in D \]

Let us denote by \( S(a) \) the largest number \( \lambda \) such that

(6.9) holds. One gets

(6.10) \[ S(a) > 0, \quad S(e) = 1 \]

Form this we can construct a pre-norm on \( \mathcal{M} \) by defining

(6.11) \[ g(a) = [S(a)S(a^*)]^{-1/2} \]
References