THE CLOSURE OF MINKOWSKI SPACE

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Recently Penrose [1] has proposed to close space-time by world points at infinity. The closure of space-time provides some progress in studying asymptotic proporties and proporties "in the large". As a rule, the metric is singular at infinity and the structure of these singularities will be a characteristic of the asymptotic behaviour of the metric and related quantities. Of course the closure of space-time by world points at infinity is not unique. Therefore for the singularities the same is true.

In the following a suitable closure \( \tilde{M} \) of Minkowski space \( M \) is given, which may be used to study asymptotical flat metrics [2]. Minkowskian coordinates are (algebraical) singular at infinity. The introduction of coordinates, regular at infinity, is a first step in resolving the mentioned singularities of the field quantities.

The closure of Minkowski space gives a compact real-analytic manifold, which is closely connected with the structure of the future tube [3]. The world points at infinity constitute a closed "light cone". The conformal group acts as a regular transitive group on the closed space-time.

To construct \( \tilde{M} \) we choose an orthochronous Lorentz frame \( \{x^i\}, x^0 = ct \), (the signature of \( M \) is \(+---+)\) and consider first the transformation

\[
\{x^i\} \rightarrow x^0 E + x^1 \sigma_1 + x^2 \sigma_2 + x^3 \sigma_3 = H
\]  

(1)

Here \( E \) is the unity matrix and the \( \sigma_\alpha \) are the Pauli matrices. We consider the matrix elements of \( H \) as new coordinates of the world points. Every world point is in one-to-one correspondence to a Hermitian matrix \( H \). Note the well known relation

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = |dH|.
\]  

(2)

Now we change coordinates once more by a Cayley transformation:

\[
H \rightarrow (H - iE)(H + iE)^{-1} = U.
\]  

(3)

Now we identify \( \tilde{M} \), which is the closure of \( M \), with the manifold of all \( 2 \times 2 \) unitary matrices, which is a (real-algebraic) manifold. As (1) and (3) give a one-to-one map of \( M \) into the set of unitary matrices, we identify \( M \) with the set of those \( U \), which are pictures of world points.
of the Minkowski space. In this way \( M \) is represented by the unitary matrices with \( |E - U| \neq 0 \) while the world points at infinity are given by \( |E - U| = 0 \). With the aid of (2) and (3) we conclude

\[
ds^2 = -4|E - U|^{-2} - 2|dU|.
\] (4)

As \( |dU| \) is a (non real) regular metric on \( \bar{M} \), \( ds^2 \) is conformally equivalent to an everywhere regular metric and has a pole of order two at infinity\(^1\). Therefore \( g^R \) is a tensor in \( \bar{M} \) vanishing at the world points at infinity. Remark that \( |E - U| = -4(\bar{x} \cdot \bar{x} - 1 + 2ix^0) \). We consider now the mappings \( U \rightarrow (AU + B)(CU + D)^{-1} \) with \( AA^* - CC^* = DD^* - BB^* = E \) and \( AB^* = CD^* \). They define a connected 15-parametric group \( \Gamma \), found by E. Cartan \(^4\), transitive on the \( 2 \times 2 \) unitary matrices and this means on \( \bar{M} \). A simple calculation shows that \( \Gamma \) consists of the conformal transformations of the metric \( |dU| \) and therefore (see Eq. 4) \( \Gamma \) is the connected component of the conformal group of the Minkowski line-element, and

acts on \( \bar{M} \) as a group of homeomorphisms without singularities\(^2\). We conclude that \( |U - E| = = 0 \) is congruent to a light-cone (with origin \( E \)). To see something about the structure of a closed light-cone, we consider the simpler cases \( x^2 = x^3 = 0 \) and \( x^2 = 0 \). In case one (only \( x^0 \) and \( x^1 \) are considered) \( \bar{M} \) is topological a torus and a closed light-cone is a system of two canonical cuts, which cross at the origin of the "cone". If \( x^3 \) is considered to be zero, things are more complicated and we construct a topological equivalent of the light-cone in the following way: We take a Klein's bottle, choose an equator on it and consider this equator to be one point (origin of the cone).

Finally we remark: a) Let \( \Gamma^\prime \) be the group of the proper Lorentz transformations and of the scalar transformations \( x^i \rightarrow \lambda x^i \). An element of \( \Gamma \) lies in \( \Gamma^\prime \) if and only if \( U = E \) is a fixed-point of it. Therefore there exists a natural homeomorphism \( \bar{M} \leftrightarrow \Gamma^\prime \) = space of right cosets from \( \Gamma \) to \( \Gamma^\prime \).

b) The transformations which are not connected with the identity are represented by \( U \rightarrow U^\ast, U, U^\prime (= U \) transposed).

REFERENCES


\(^{1}\) On the only singular point \( U = E \) of the hypersurface \( |E - U| = 0 \), the pole is of order four.

\(^{2}\) To see this more transparently by an analogy: If Gaussian plane is completed by "the point at infinity", the transformations \( x^i = (ax + b)(cx + d)^{-1} \) become a group of homeomorphisms.