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On Bures Distance and $*$-Algebraic Transition Probability between Inner Derived Positive Linear Forms over W*-Algebras

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Abstract. On a W*-algebra M, for given two positive linear forms $\nu, \varrho \in M_+^*$ and algebra elements $a, b \in M$, a variational expression for the Bures distance $d_{\mathrm{B}}(\nu^a, \varrho^b)$ between the inner derived positive linear forms $\nu^a = \nu(a^* \cdot a)$ and $\varrho^b = \varrho(b^* \cdot b)$ is obtained. Along with the proof of the formula, also an earlier result of S. Gudder on noncommutative probability will be slighly extended. Also, the given expression of the Bures distance relates nicely to the system of seminorms proposed by D. Buchholz which occurs, along with the problem of estimating the so-called 'weak intertwiners', in algebraic quantum field theory. In the last section, some optimization problem will be considered.

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1. Introduction

1.1. BASIC SETTINGS ON BURES DISTANCE

Throughout the paper, the Bures distance function d_{B} [11] and related metric concepts on the positive cone M_+^* of the bounded linear forms M^* over a W*-algebra M will be considered. We start by defining the Bures distance $d_{\mathrm{B}}(M|\nu, \varrho)$ between $\nu, \varrho \in M_+^*$.

DEFINITION 1. $d_{\mathrm{B}}(M|\nu, \varrho) = \inf_{\{\pi, \mathcal{K}\}, \varphi \in \mathcal{S}_{\pi,M}(\nu), \psi \in \mathcal{S}_{\pi,M}(\varrho)} \|\psi - \varphi\|$.

Instead of $d_{\mathrm{B}}(M|\nu, \varrho)$ $d_{\mathrm{B}}(\nu, \varrho)$ will often be used. For unital $*$-representation $\{\pi, \mathcal{K}\}$ of M on a Hilbert space $\{\mathcal{K}, \langle \cdot, \cdot \rangle\}$ and for $\mu \in M_+^*$, we let

$$\mathcal{S}_{\pi,M}(\mu) = \{\xi \in \mathcal{K} : \mu(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle\}.$$

Then, the infimum within the defining formula for $d_{\mathrm{B}}(\nu, \varrho)$ extends over all those π relative to which $\mathcal{S}_{\pi,M}(\nu) \neq \emptyset$ and $\mathcal{S}_{\pi,M}(\varrho) \neq \emptyset$ simultaneously hold and, within

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each such representation, the vectors φ and ψ may be varied through all of $\mathcal{S}_{\pi,M}(\nu)$ and $\mathcal{S}_{\pi,M}(\varrho)$, respectively. The scalar product $\mathcal{K} \times \mathcal{K} \ni \{\chi, \eta\} \longmapsto \langle\chi, \eta\rangle \in \mathbb{C}$ on the representation Hilbert space by convention is supposed to be linear with respect to the first argument χ, antilinear in the second argument η, and maps into the complex field $\mathbb{C}$. Let $\mathbb{C} \ni z \mapsto \bar{z}$ be the complex conjugation and $\Re z$ and $|z|$ be the real part and absolute value of z, respectively. Greek letters and their labelled derivates (except for π, which is reserved for representations only) will be used to label elements of the complex Hilbert spaces on which the concrete C^*-algebras $\pi(M)$ are supposed to act. The norm of $\chi \in \mathcal{K}$ is given by $\|\chi\| = \sqrt{\langle\chi, \chi\rangle}$. For the relating operator and C^*-algebra theory, the reader is referred to the standard monographs, e.g. [13, 19, 23].

For simplicity, for the C^*-norm of an element $x \in M$ as well as for the operator norm of an concrete bounded linear operator $x \in \mathsf{B}(\mathcal{K})$, the same notation $\|x\|$ will be used. In both these cases, the involution ($*$-operation), respectively, the taking of the hermitian conjugate of an element x, is indicated by the transition $x \longmapsto x^*$. The notions of Hermiticity and positivity for elements are defined as usual in C^*-algebra theory, and M_{h} and M_+ are the Hermitian and positive elements of M, respectively. In view to the above, and to make these settings more unambiguous, Greek letters will *not* be used as symbols for linear operators over $\mathcal{K}$ or elements of M. The null and the unit element/operator in M and $\mathsf{B}(\mathcal{K})$ will be denoted by $\mathbf{0}$ and $\mathbf{1}$.

For notational purposes generally, we recall some fundamentals relating (bounded) linear forms which subsequently might be of concern within the context of Definition 1. Recall that the topological dual space M^* of M is the set of all those linear functionals (linear forms) which are continuous with respect to the operator norm topology. Equipped with the dual norm $\|\cdot\|_1$, which is given by $\|f\|_1 = \sup\{|f(x)| : x \in M,\ \|x\| \leqslant 1\}$ and which is referred to as the functional norm, M^* is a Banach space. For each given $f \in M^*$, the Hermitian conjugate functional $f^* \in M^*$ is defined by $f^*(x) = \overline{f(x^*)}$, for each $x \in M$. Note that $f \in M^*$ is Hermitian if $f = f^*$ holds and f is termed positive if $f(x) \geqslant 0$ holds, for each $x \in M_+$. Also remember that a bounded linear form over M is positive if and only if $\|f\|_1 = f(\mathbf{1})$ is fulfilled. For positive linear forms, one has the following fundamental estimate (Cauchy–Schwarz inequality):

$$\forall\, g \in M^*_+ : \ |g(y^*x)|^2 \leqslant g(y^*y)\, g(x^*x), \quad \forall\, x, y \in M\,, \tag{1.1a}$$

which, accordingly, also holds on C^*-algebras. From this, it is easily inferred that for each $g \in M^*_+ \backslash \{\mathbf{0}\}$ the subset $I_g \subset M$ defined by

$$I_g = \{x \in M : \ g(x^*x) = 0\} \tag{1.1b}$$

is a (proper) *left ideal* in M. Provided this ideal is trivial, $I_g = \{\mathbf{0}\}$, the positive linear form $g \in M^*_+$ is called *faithful* (positive linear form).

The most important consequence of positivity and (1.1a) is that, for each $g \in M^*_+$, there exists a cyclic $*$-representation π_g of M on some Hilbert space $\mathcal{K}_g$, with cyclic vector $\Omega \in \mathcal{K}_g$, and obeying $g(x) = \langle \pi_g(x)\Omega, \Omega\rangle$, for all $x \in M$. This fact is usually referred to as the Gelfand–Neumark–Segal theorem (GNS). Such a representation (which is unique up to unitary isomorphisms) will be referred to as a g associated cyclic representation or a GNS-representation of g, respectively. Note that considering such a construction in the special case with $g = \nu + \varrho$ will provide a unital $*$-representation $\pi = \pi_g$ such that $\mathcal{S}_{\pi,M}(\nu) \neq \emptyset$ and $\mathcal{S}_{\pi,M}(\varrho) \neq \emptyset$ hold (we omit the details, all of which are standard). It is exactly this fact which makes the expression in Definition 1 make sense even in the C^*-algebraic case.

Apart from the functional norm topology, we also mention the w^*-topology on M^*, which is the weakest locally convex topology generated by the seminorms ρ_x, $x \in M$, with $\rho_x(f) = |f(x)|$, for each $f \in M^*$. Recall that, according to basic result of Banach space theory (the Alaoglu–Banach theorem), each closed, bounded subset of the dual Banach space M^* has to be w^*-compact.

Along with Definition 1, an auxiliary metric structure arises which can be compared to the metric structure given by the 'natural' distance $d_1(\nu, \varrho) = \|\nu - \varrho\|_1$ on M^*_+. The relevant basic facts will be stated here without proof and read as follows:

PROPOSITION 1. *Let* $d_{\mathrm{B}}\colon M^*_+ \times M^*_+ \ni \{\nu, \varrho\} \longmapsto d_{\mathrm{B}}(M|\nu, \varrho) \in \mathbb{R}_+$ *be given in accordance with Definition* 1. *Then the following hold*:

(1) d_{B} *is a distance function on the points of* M^*_+*;*
(2) d_{B} *is topologically equivalent to* d_1 *on bounded subsets of* M^*_+.

Especially for $\{\nu, \varrho\} \in M^*_+ \times M^*_+ \backslash \{0, 0\}$, *one has*

$$c(\nu, \varrho)^{-1}\, d_1(\nu, \varrho) \leqslant d_{\mathrm{B}}(M|\nu, \varrho) \leqslant \sqrt{d_1(\nu, \varrho)}\,, \tag{1.2}$$

with $c(\nu, \varrho) = \sqrt{\|\nu\|_1} + \sqrt{\|\varrho\|_1}$.

Remark that item (1) and 'one half' of the estimate (1.2), from which (2) obviously can be followed, were anticipated and proved by D. Bures in [11], whereas the other half of (1.2) can be seen by arguments given by H. Araki in [6, 7], e.g. omit any details on this matter but remark that D. Bures refers to the *state space* of M, $\mathcal{S}(M) = \{f \in M^*_+ : f(\mathbf{1}) = 1\}$. This simplifies matters insofar that, in restriction to $\mathcal{S}(M)$, d_{B} gets an unconditionally topologically equivalent with d_1.

1.2. PREREQUISITES, USEFUL ESTIMATES AND EXAMPLES

In conjunction with the Bures distance d_{B}, one has the functor P of the ($*$-algebraic) *transition probability* [25]. For given W^*-algebra M and positive linear forms $\nu, \varrho \in M^*_+$, the definition reads as follows:

DEFINITION 2. $P_M(\nu, \varrho) = \sup_{\{\pi, \mathcal{K}\}, \varphi \in \mathcal{S}_{\pi,M}(\nu), \psi \in \mathcal{S}_{\pi,M}(\varrho)} |\langle \psi, \varphi\rangle|^2\,.$

Thereby, the range of variables over which the supremum has to be extended is the same as in Definition 1. With the help of P_M, one then gets a (uniquely solvable) expression for the Bures distance:

$$d_{\mathrm{B}}(M|\nu,\varrho)^2 = \left\{\|\nu\|_1 - \sqrt{P_M(\nu,\varrho)}\right\} + \left\{\|\varrho\|_1 - \sqrt{P_M(\nu,\varrho)}\right\}. \tag{1.3}$$

Remark that P is of importance in its own right (and independent of the aforementioned appearance within (1.3)) since it can be easily adapted to several applications in (algebraic) quantum physics, non-commutative probability and estimation theory. The latter also was the heuristic intention behind the introduction of this functor in [25]. For a particular range of applications see, e.g., [4, 27].

Many properties of P are known. In the following, only a few of these properties will be explicitly referred to. For instance, essentially, by means of the Cauchy–Schwarz inequality from the definition of P, the following fundamental estimates can be obtained:

$$|f(\mathbf{1})|^2 \leqslant P_M(\nu,\varrho) \leqslant \nu(a)\,\varrho(a^{-1}), \tag{1.4}$$

where f can be any linear form of the set

$$\Gamma_M(\nu,\varrho) = \left\{f \in M^* : \left|f(y^*x)\right|^2 \leqslant \nu(y^*y)\varrho(x^*x),\ \forall\, x, y \in M\right\} \tag{1.5}$$

and a can be any invertible, positive element $a \in M_+$. Note that $\Gamma_M(\nu,\varrho)$ is obviously w^*-closed and bounded ($\sqrt{\|\nu\|_1\|\varrho\|_1}$ is a common upper bound), and thus is a w^*-compact subset of M^*.

For the estimate from above see equation (16) in [25]. Relating the estimate from below, suppose that a unital $*$-representation $\{\pi, \mathcal{K}\}$ of M on $\mathcal{K}$ with $\mathcal{S}_{\pi,M}(\nu) \neq \emptyset$ and $\mathcal{S}_{\pi,M}(\varrho) \neq \emptyset$ is given. From the standard facts, one then infers that for given $\varphi \in \mathcal{S}_{\pi,M}(\nu)$, $\psi \in \mathcal{S}_{\pi,M}(\nu)$,

$$\Gamma_M(\nu,\varrho) = \left\{\langle \pi(\cdot)k\psi, \varphi\rangle \,:\, k \in (\pi(M)')_1\right\} \tag{1.6}$$

has to be fulfilled. In this formula $(\pi(M)')_1$ is the unit ball within the commutant vN-algebra $\pi(M)'$. From this and Definition 2 and with the help of the Theorem of B. Russo and H. Dye [15], the validity of the estimate from below in (1.4) also follows, see equation (3) in [1].

Apply (1.4) to the special case of two vector states, which is heuristically important in a quantum physical context of two wave functions:

EXAMPLE 1. Let $M = \mathsf{B}(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Let $\mu_\psi = \langle(\cdot)\psi, \psi\rangle$ be the vector form generated by $\psi \in \mathcal{H}$ on M, and be p_φ the orthoprojection onto the span of $\varphi \in \mathcal{H}$. Then, considering $f = \langle(\cdot)\psi, \varphi\rangle \in \Gamma_M(\mu_\varphi, \mu_\psi)$ and $a = p_\varphi + \varepsilon^{-1}p_\varphi^\perp$, for $\varepsilon \in \mathbb{R}_+\backslash\{0\}$, and inserting this into (1.4) provides $|\langle\psi,\varphi\rangle|^2 \leqslant P_M(\mu_\varphi,\mu_\psi) \leqslant |\langle\psi,\varphi\rangle|^2 + \varepsilon\|\varphi\|^2\mu_\psi(p_\varphi^\perp)$. From this,

$$P_M(\mu_\varphi,\mu_\psi) = |\langle\psi,\varphi\rangle|^2$$

follows for $\varepsilon \to 0$, in any case of two vectors $\psi, \varphi \in \mathcal{H}$.

Also, constellations among the positive linear forms $\nu, \varrho \in M_+^*$ are known such that, for some $a \geqslant \mathbf{0}$, the upper estimate within (1.4) turns into an equality. This then provides an expression for $P_M(\nu, \varrho)$.

To explain such stuff, fix some notation first. In all that follows for $x \in M$ and $\mu \in M_+^*$, a positive linear form μ^x will be defined by $\mu^x(y) = \mu(x^* y x)$, for each $y \in M$. If this situation occurs, the positive linear form μ^x will be referred to as an *inner derived* (from μ) positive linear form. The main result of [25] refers to this and reads as follows:

THEOREM 1. $\forall\, \mu \in M_+^*,\ a, b \in M,\ a^* b \geqslant \mathbf{0} \,:\, \sqrt{P_M(\mu^a, \mu^b)} = \mu(a^* b)$.

For instance, in choosing $a \geqslant \mathbf{0},\ b = \mathbf{1}$, the premises of the previous result are fulfilled in a trivial manner and one thus arrives at the formula

$$\forall\, \mu \in M_+^*,\ a \in M_+ \,:\, \quad P_M(\mu^a, \mu) = \mu(a)^2\,. \tag{1.7}$$

Remark that Example 1 in the case of nonorthogonal vectors can be seen as a special case of (1.7) as well. It is interesting that the seemingly very special situation with the premises of (1.7) addresses itself to a wide range of characteristic applications. One of these reads as follows:

EXAMPLE 2. By the Radon–Nikodym theorem of S. Sakai [22] we are always in such a situation if, amongst two *normal* positive linear forms $\nu, \varrho \in M_+^*$, a relation of domination $\varrho \leqslant \lambda\nu$, with $\lambda \in \mathbb{R}_+\backslash\{0\}$, takes place, in which situation the notation $\varrho \ll \nu$ will be also used. That is, for $\varrho \ll \nu$, there is $a \in M_+$ with $\varrho = \nu^a$. In view of the above, a in such situation, $P_M(\varrho, \nu) = \nu(a)^2$ especially follows. It is known that a becomes unique if $s(a) \leqslant s(\nu)$ is required to hold, with the support of the operator a and the normal positive linear form ν, respectively. One usually refers to this unique a as Sakai's Radon–Nikodym operator of ϱ relative to ν, and then also the notation $a = \sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}$ will be used.

Finally, it is interesting that, in any case with the help of the bounds appearing along with (1.4), the value of $P_M(\nu, \varrho)$ can be approximated to an arbitrary degree of precision from both sides. This and some other relevant information will be the content of the following result.

THEOREM 2. *Let M be a* W*-*algebra, and be $\nu, \varrho \in M_+^*$. Then, the following facts hold*:

(1) $\sqrt{P_M(\nu, \varrho)} = \inf_{x > \mathbf{0}} \sqrt{\nu(x)\varrho(x^{-1})}$;
(2) $\sqrt{P_M(\nu, \varrho)} = \sup_{f \in \Gamma_M(\nu,\varrho)} |f(\mathbf{1})|\,$.

The infimum in (1) *extends over all positive invertible elements of M. Moreover, if $\{\pi, \mathcal{K}\}$ is any unital $*$-representation of M over some Hilbert space $\mathcal{K}$ such that $\mathcal{S}_{\pi,M}(\nu) \neq \emptyset$ and $\mathcal{S}_{\pi,M}(\varrho) \neq \emptyset$ are fulfilled, then the following is fulfilled*:

(3) $\sqrt{P_M(\nu, \varrho)} = \sup_{\psi \in \mathcal{S}_{\pi,M}(\varrho)} |\langle \psi, \varphi \rangle| \,, \ \forall \varphi \in \mathcal{S}_{\pi,M}(\nu) \,.$

Also, the supremum in (2) *is a maximum and is attained at some* $f \in \Gamma_M(\nu, \varrho)$*, and some maximizing* f *can be chosen as* $f = \langle \pi(\cdot)\psi_0, \varphi_0 \rangle$*, for some* $\psi_0 \in \mathcal{S}_{\pi,M}(\varrho)$, $\varphi_0 \in \mathcal{S}_{\pi,M}(\nu)$.

For proofs of (1)–(3) see Corollary 1, Corollary 3 and Theorem 3 in [1], for the additional informations on the attainability of the supremum in (2), see [7] and [2]. The previous result remains valid even if M is supposed to be a unital C^*-algebra.

Remark 1. The question arises whether the functor P in a reasonable manner (i.e. such that a relation of type (1.3) with a metric distance d_{B} remained true on its domain of definition) could be extended to some yet more general category of $*$-algebras (including some unbounded operator algebras showing up in relativistic quantum field theory for example), see [24, 26]. Besides the just-mentioned C^*-algebraic cases, the answer seems to be in the negative.

1.3. THE MAIN RESULT

Under the premises of Theorem 1, let us suppose now that some unital $*$-representation $\{\pi, \mathcal{K}\}$ has been chosen in accordance with $\mathcal{S}_{\pi,M}(\mu) \neq \emptyset$. Then, for $\Omega \in \mathcal{S}_{\pi,M}(\mu)$, one has $\pi(a)\Omega \in \mathcal{S}_{\pi,M}(\mu^a)$ and $\pi(b)\Omega \in \mathcal{S}_{\pi,M}(\mu^b)$. Hence, in making use of (1.6) in the special case of $\Gamma_M(\mu, \mu)$, with $\varphi = \psi = \Omega$, and in the special case of $\Gamma_M(\mu^a, \mu^b)$ with $\psi = \pi(b)\Omega$ and $\varphi = \pi(a)\Omega$, and respecting the positivity of a^*b, one easily infers that

$$\mu(a^*b) = \left\|\pi(\sqrt{a^*b})\Omega\right\|^2 = \sup_{g \in \Gamma_M(\mu,\mu)} |g(a^*b)| = \sup_{f \in \Gamma_M(\mu^a,\mu^b)} |f(\mathbf{1})|$$

has to be fulfilled. The formula of Theorems 1 and 2(2), together with the previous result, then show that the following is valid:

COROLLARY 1.

$$\forall \mu \in M_+^*, \ a, b \in M, \ a^*b \geqslant \mathbf{0} \ : \ \sqrt{P_M(\mu^a, \mu^b)} = \sup_{f \in \Gamma_M(\mu,\mu)} |f(a^*b)| \,.$$

The first goal of the paper will be to extend the assertion of Corollary 1 to hold true under much weaker premises. More precisely, instead of considering two positive linear forms ν, ϱ which are both inner derived positive linear forms $\nu = \mu^a$ and $\varrho = \mu^b$ from one and the same positive linear form μ via operators $a, b \in M$, which obey the positivity assumption $a^*b \geqslant \mathbf{0}$, subsequently two arbitrarily chosen inner derived positive linear forms are permitted to be considered without any further restriction. Based on this a variational expression for the Bures distance function will be derived, under the same premises as the positive linear forms.

THEOREM 3. *Let M be a* W^**-algebra, and be $\nu, \varrho \in M_+^*$, and $a, b \in M$. Then, the following facts hold true*:

(1) $\sqrt{P_M\bigl(\nu^a, \varrho^b\bigr)} = \sup_{f \in \Gamma_M(\nu,\varrho)} |f(a^*b)|$;
(2) $d_{\mathrm{B}}(M|\nu^a, \varrho^b)^2 = \sup_{a^*b=y^*x} \bigl\{\nu(a^*a - y^*y) + \varrho(b^*b - x^*x)\bigr\}$.

Obviously, (1) is the announced extension of the assertion of Corollary 1, whereas by (2), which will be shown to be a consequence of (1), the mentioned variational expression for the distance d_{B} between the two inner positive linear forms derived from a given pair $\{\nu, \varrho\}$ is given.

Foremost, such an expression as given in (2) can be useful since it allows for estimating the behavior of the Bures distance at $\{\nu, \varrho\}$ if this pair is undergoing an inner perturbation towards another pair $\{\nu^a, \varrho^b\}$ of positive linear forms. As it comes out, the geometry of submanifolds of mutually coordinated (via inner operations) positive linear forms of W^*-algebras of use to us, should be based on this formula. We will not elaborate on this in this paper, but instead we will be concerned with one particular aspect of this geometry more in detail within Section 3.

In the course of the derivation of the main result, several other characterizations of P (and thus of d_{B} as well) will be obtained.

2. Results and Proofs

2.1. FURTHER CHARACTERIZATIONS OF TRANSITION PROBABILITY

In all what follows⋆ M is a W^*-algebra and $\nu, \varrho \in M_+^*$ are fixed but can be arbitrarily chosen positive linear forms. We start with some consequences from Theorem 2. Relating notations, when occurring in conjunction with inf or sup, in each case of occurrence, the variables $x > \mathbf{0}$, $\{x\}$, $\{e\}$ and $\{y, x\}$ are thought to extend over all positive invertible elements x, all finite decompositions $\{x\} = \{x_1, \ldots, x_n\}$ of the unity into positive elements, all finite decompositions $\{e\} = \{e_1, \ldots, e_n\}$ of the unity into orthoprojections, and all finite double systems $\{y, x\} = \{y_1, x_1, \ldots, y_n, x_n\}$ of elements obeying $\sum_j y_j^* x_j = \mathbf{1}$, respectively, within M, where n can range through the naturals, $n \in \mathbb{N}$.

COROLLARY 2. *The following properties hold*:

(1) $\sqrt{P_M(\nu, \varrho)} = \inf_{\{x\}} \sum_j \sqrt{\nu(x_j)\varrho(x_j)}$;
(2) $\sqrt{P_M(\nu, \varrho)} = \inf_{\{e\}} \sum_j \sqrt{\nu(e_j)\varrho(e_j)}$;

⋆ Most of the material of Section 2 as well as some parts of Section 3, especially 3.2, are reproduced from the part 'foundational material' of the manuscript [3].

(3) $\sqrt{P_M(\nu,\varrho)} = \inf_{\{y,x\}} \frac{1}{2} \sum_j \left\{\nu(y_j^* y_j) + \varrho(x_j^* x_j)\right\}$;
(4) $\sqrt{P_M(\nu,\varrho)} = \inf_{\{\mathbf{1}=y^*x\}} \frac{1}{2} \left\{\nu(y^* y) + \varrho(x^* x)\right\}$;
(5) $\sqrt{P_M(\nu,\varrho)} = \inf_{x>\mathbf{0}} \frac{1}{2} \left\{\nu(x) + \varrho(x^{-1})\right\}$.

Proof. Note that according to (1.5) for each $f \in \Gamma_M(\nu,\varrho)$ and any finite positive decomposition $\{x\}$ of the unity, one has

$$|f(\mathbf{1})| \leqslant \sum_j |f(x_j)| = \sum_j |f(\sqrt{x_j}\sqrt{x_j})| \leqslant \sum_j \sqrt{\nu(x_j)\varrho(x_j)}.$$

According to Theorem 2(2),

$$\sqrt{P_M(\nu,\varrho)} \leqslant \inf_{\{x\}} \sum_j \sqrt{\nu(x_j)\varrho(x_j)} \leqslant \inf_{\{e\}} \sum_j \sqrt{\nu(e_j)\varrho(e_j)} \tag{$\star$}$$

can be followed. That is, the validity of (2) will imply that (1) is also true. To see that (2) holds, let $\varepsilon > 0$. According to Theorem 2(1), there exists invertible $x \in M_+$ obeying $\nu(x)\varrho(x^{-1}) < P_M(\nu,\varrho) + \varepsilon$. Since the map $y \longmapsto y^{-1}$, in restriction to the invertible elements of M_+, is normcontinuous, and since we are in a W*-algebra, we may additionally suppose that x satisfying the above estimate is chosen with a finite spectrum, that is, $x = \sum_{j=1}^n \lambda_j e_j$ is fulfilled with $\lambda_j > 0$, and some finite decomposition $\{e_1, \ldots, e_n\}$ of the unity into mutually orthogonal orthoprojections of M. Using this spectral decomposition, one arrives at the expression

$$\begin{aligned}&\nu(x)\varrho(x^{-1})\\ &\quad = \sum_j \nu(e_j)\varrho(e_j) + \sum_{j>k}\left\{\lambda_j\lambda_k^{-1}\nu(e_j)\varrho(e_k) + \lambda_k\lambda_j^{-1}\nu(e_k)\varrho(e_j)\right\}.\end{aligned}$$

Owing to the strict positivity of the λ's and the nonnegativity of the $\nu(e_j)$'s, one has

$$\begin{aligned}&\lambda_j\lambda_k^{-1}\nu(e_j)\varrho(e_k) + \lambda_k\lambda_j^{-1}\nu(e_k)\varrho(e_j)\\ &\quad \geqslant 2\sqrt{\nu(e_j)\varrho(e_j)}\sqrt{\nu(e_k)\varrho(e_k)}\end{aligned}$$

for each $j > k$. In fact, this is trivial for $\sqrt{\nu(e_j)\varrho(e_j)}\sqrt{\nu(e_k)\varrho(e_k)} = 0$, whereas in the other case, the estimate follows from minimizing the positive function $F(t) = t\,\nu(e_j)\varrho(e_k) + t^{-1}\,\nu(e_k)\varrho(e_j)$ over $\mathbb{R}_+\backslash\{0\}$, which has a solution, since in this case both coefficients of t and t^{-1} are strictly positive. By means of this estimate and the above, one finally arrives at

$$P_M(\nu,\varrho) + \varepsilon \geqslant \nu(x)\varrho(x^{-1}) \geqslant \left\{\sum_j \sqrt{\nu(e_j)\varrho(e_j)}\right\}^2. \tag{$\star\star$}$$

From this

$$\inf_{\{p\}} \sum_j \sqrt{\nu(p_j)\varrho(p_j)} \leqslant \sqrt{P_M(\nu,\varrho) + \varepsilon}$$

is seen. Since $\varepsilon > 0$ could have been chosen at will, $\sqrt{P_M(\nu,\varrho)} \geqslant \inf_{\{p\}} \sum_j \sqrt{\nu(p_j)\varrho(p_j)}$ follows with $\{p\}$ extending over the finite decompositions of the unity into orthoprojections of M. From this and ($\star$), follow (1) and (2).

In order to prove (3), to given $\varepsilon > 0$, for each $\delta > 0$ by means of the decomposition $\{e_1,\ldots,e_n\}$ of the unity into orthoprojections e_j obeying ($\star\star$) let us define a double system $\{y(\delta),x(\delta)\} \subset M$ by setting $x_j(\delta)=\mu_j(\delta)\,e_j$, $y_j(\delta)=\mu_j(\delta)^{-1}\,e_j$, with

$$\mu_j(\delta)=\sqrt[4]{\frac{\nu(e_j)+\delta}{\varrho(e_j)+\delta}}$$

for each $j \leqslant n$. Then, also $\sum_j y_j^*(\delta)x_j(\delta)=\mathbf{1}$ holds, and therefore the double system $\{y(\delta),x(\delta)\}$ is a special case of those double systems considered within the context of the infimum in (3). Hence, one has $\frac{1}{2}\inf_{\{y,x\}}\sum_j \big\{\nu(y_j^*y_j)+\varrho(x_j^*x_j)\big\} \leqslant F(\delta)$, for each $\delta > 0$, with the auxiliary function $\delta \mapsto F(\delta)$ defined by

$$F(\delta)=\frac{1}{2}\sum_j \big\{\nu(y_j(\delta)^*y_j(\delta))+\varrho(x_j(\delta)^*x_j(\delta))\big\}.$$

Since with this choice, one easily infers that $F(\delta)$ may be expressed as

$$\begin{aligned} F(\delta) \;=\; & \sum_{j,\,\nu(e_j)\neq 0} \frac{1}{2}\sqrt{\{\varrho(e_j)+\delta\}\nu(e_j)}\,\sqrt{\frac{\nu(e_j)}{\nu(e_j)+\delta}}\;+ \\ & +\sum_{j,\,\varrho(e_j)\neq 0} \frac{1}{2}\sqrt{\{\nu(e_j)+\delta\}\varrho(e_j)}\,\sqrt{\frac{\varrho(e_j)}{\varrho(e_j)+\delta}}\,, \end{aligned}$$

in view of the previous and ($\star\star$), then

$$\lim_{\delta\to 0} F(\delta)=\sum_j \sqrt{\nu(e_j)\varrho(e_j)} \leqslant \sqrt{P_M(\nu,\varrho)+\varepsilon} \tag{$\star'$}$$

can be followed. Therefore

$$\sqrt{P_M(\nu,\varrho)+\varepsilon} \geqslant \frac{1}{2}\inf_{\{y,x\}}\sum_j \big\{\nu(y_j^*y_j)+\varrho(x_j^*x_j)\big\}$$

is seen. Since such a procedure can be performed for each $\varepsilon > 0$, one can be assured that

$$\sqrt{P_M(\nu,\varrho)} \geqslant \frac{1}{2}\inf_{\{y,x\}}\sum_j \big\{\nu(y_j^*y_j)+\varrho(x_j^*x_j)\big\}$$

is fulfilled, where $\{y,x\}$ is allowed to run through all finite double systems obeying $\sum_j y_j^*x_j=\mathbf{1}$. On the other hand, for each such double system and $f \in \Gamma_M(\nu,\varrho)$, one has

$$|f(\mathbf{1})| \leqslant \sum_j |f(y_j^*x_j)| \leqslant \sum_j \sqrt{\nu(y_j^*y_j)\varrho(x_j^*x_j)}\,.$$

Now, for each two elements $x, y \in M$, the estimate $\sqrt{\nu(y^*y)\varrho(x^*x)} \leqslant \frac{1}{2}\{\nu(y^*y)+\varrho(x^*x)\}$ is inferred from $\{\sqrt{\nu(y^*y)}-\sqrt{\varrho(x^*x)}\}^2 \geqslant 0$. Hence, the above estimate relating double systems can be continued accordingly and results in $|f(\mathbf{1})| \leqslant \frac{1}{2}\sum_j\{\nu(y_j^*y_j)+\varrho(x_j^*x_j)\}$. This has to hold for each $f \in \Gamma_M(\nu,\varrho)$ and finite double system $\{y,x\}$ obeying $\sum_j y_j^*x_j = \mathbf{1}$. Thus, also

$$\sqrt{P_M(\nu,\varrho)} \leqslant \frac{1}{2}\inf_{\{y,x\}}\sum_j\left\{\nu(y_j^*y_j)+\varrho(x_j^*x_j)\right\}$$

is seen. In view of the above, equality follows, that is, (3) is seen to hold. Note that within the context of ($\star'$) if an element $a(\delta) \in M$ is defined by means of the above $y_j(\delta)$ through the setting $a(\delta) = \sum_j y_j(\delta)^* y_j(\delta)$, one has $a(\delta) > \mathbf{0}$, invertible with $a(\delta)^{-1} = \sum_j x_j(\delta)^* x_j(\delta)$, and then ($\star'$) under the above premises on ε equivalently also shows that

$$\lim_{\delta\to 0}\frac{1}{2}\left\{\nu((a(\delta))+\varrho(a(\delta)^{-1})\right\} = \sum_j\sqrt{\nu(e_j)\varrho(e_j)} \leqslant \sqrt{P_M(\nu,\varrho)+\varepsilon}$$

has to be fulfilled. Since $\varepsilon > 0$ can be arbitrarily chosen, from the previous, then even an estimate

$$\sqrt{P_M(\nu,\varrho)} \geqslant \inf_{x>\mathbf{0}}\frac{1}{2}\left\{\nu(x)+\varrho(x^{-1})\right\} \tag{$\star''$}$$

can be seen to be fulfilled, where now the infimum extends over all invertible, positive elements of M. On the other hand, for each invertible, positive element $x \in M$, one has the identity

$$\begin{aligned}&\frac{1}{2}\left\{\sqrt{\nu(x)}-\sqrt{\varrho(x^{-1})}\right\}^2+\sqrt{\nu(x)\varrho(x^{-1})}\\ &\quad=\frac{1}{2}\left\{\nu(x)+\varrho(x^{-1})\right\}.\end{aligned} \tag{2.1a}$$

Taking the infimum over the invertible positive $x \in M$ on both sides and respecting the nonnegativity of $(1/2)\{\sqrt{\nu(x)}-\sqrt{\varrho(x^{-1})}\}^2$, will show that the following estimate has to be fulfilled:

$$\begin{aligned}\inf_{x>\mathbf{0}}\sqrt{\nu(x)\varrho(x^{-1})} &\leqslant \inf_{x>\mathbf{0}}\frac{1}{2}\left\{\sqrt{\nu(x)}-\sqrt{\varrho(x^{-1})}\right\}^2+\inf_{x>\mathbf{0}}\sqrt{\nu(x)\varrho(x^{-1})}\\ &\leqslant \inf_{x>\mathbf{0}}\frac{1}{2}\left\{\nu(x)+\varrho(x^{-1})\right\}.\end{aligned} \tag{2.1b}$$

Hence, from Theorem 2(1) one can conclude that $\sqrt{P_M(\nu,\varrho)} \leqslant \inf_{x>\mathbf{0}}(1/2)\left\{\nu(x)+\varrho(x^{-1})\right\}$ has to hold. From this, in view of ($\star''$), the validity of (5) follows.

Finally, for each $\varepsilon > 0$ by the proof of (5) there exists an invertible $a > \mathbf{0}$ obeying $\sqrt{P_M(\nu,\varrho)}+\varepsilon \geqslant (1/2)\left\{\nu(a)+\varrho(a^{-1})\right\}$. In defining $y_\varepsilon = \sqrt{a}$ and $x_\varepsilon = \sqrt{a}^{-1}$, one has $\mathbf{1} = y_\varepsilon^* x_\varepsilon$, and the above estimate then turns into

$$(1/2)\left\{\nu(y_\varepsilon^*y_\varepsilon)+\varrho(x_\varepsilon^*x_\varepsilon)\right\} \leqslant \sqrt{P_M(\nu,\varrho)}+\varepsilon.$$

On the other hand, according to (3), one has

$$\sqrt{P_M(\nu,\varrho)} \leqslant \inf_{\{\mathbf{1}=y^*x\}} (1/2)\big\{\nu(y^*y)+\varrho(x^*x)\big\} \leqslant (1/2)\big\{\nu(y_\varepsilon^* y_\varepsilon)+\varrho(x_\varepsilon^* x_\varepsilon)\big\}\,.$$

From these estimates, and since $\varepsilon > 0$ can be taken at will, the validity of (4) becomes evident. This completes the proof of all the assertions.

2.2. MISCELLANEOUS COMMENTS

In the following, we will comment on the facts arising from Corollary 2, and will supplement them with further useful auxiliary results and remarks.

2.2.1. *Comments on Corollary 2(1)–(2): Quadratic Means*

For normal states, $P_M(\nu,\varrho)$ is the same as the generalized transition probability $T_M(\nu,\varrho)$ given in [12].

The definition of V. Cantoni refers to the two probability measures $\nu(E_x(\mathrm{d}\lambda))$ and $\varrho(E_x(\mathrm{d}\lambda))$ over the Borel sets of $\mathbb{R}^1$ that can be naturally associated with two normal states $\nu,\ \varrho$ on M through the projection valued measure $E_x(\mathrm{d}\lambda)$ of a self-adjoint element, say $x\in M$, with spectral representation $x=\int_{\mathbb{R}^1}\lambda\, E_x(\mathrm{d}\lambda)$ (recall that within a quantum mechanical context the Hermitian elements are the candidates of bounded observables). In line with a proposal of G. Mackey, see Chapter 2, 2.2, 2.6 in [20] and, in accordance with some physically motivated axioms saying what properties of a 'transition probability' should be considered as indispensable at all, see [21, 17, 16], e.g. in [12] one defines a generalized transition probability by

$$T_M(\nu,\varrho) = \inf_{x\in M_{\mathrm{h}}} \left\{ \int_{\mathbb{R}^1} \mathsf{QM}_x(\nu,\varrho)(d\lambda) \right\}^2 , \tag{2.2}$$

with the quadratic means

$$\mathsf{QM}_x(\nu,\varrho)(\mathrm{d}\lambda) = \sqrt{\nu(E_x(\mathrm{d}\lambda))\varrho(E_x(\mathrm{d}\lambda))}$$

of these measures, which is a Borel measure on the line again. On carefully analyzing the quadratic means in the special case of two normal states, one of which is at least faithful, the proof that $P_M(\nu,\varrho)$ of Definition 2 equals the expression (2.2) has been given in [8].

As has been remarked on by S. Gudder (see Theorem 1 in [16]), mathematically (2.2) amounts to $\sqrt{T_M(\nu,\varrho)} = \inf_{\{e\}} \sum_j \sqrt{\nu(e_j)\varrho(e_j)}$, which is (2) in this special case.

In summarizing, the information obtained through Corollary 2 on that subject is the following:

– the expression in Corollary 2(2) reflects those aspects behind (2.2) which remain valid for *any* positive linear forms (not only normal ones) on a W^*-algebra;
– the expression in Corollary 2(1) can be taken as the common general C^*-algebraic essence of the matter around quadratic means.

2.2.2. *Comments on Corollary 2(3)–(4): Some Seminorms on M*

For normal states, (3) had been conjectured by D. Buchholz, motivated by an application to relativistic quantum field theory, and has been proved in the special case of $\mathsf{B}(\mathcal{H})$ in [10], eq. (2.10). But note that there the intention was to deal with certain vector states of some $*$-algebras of (unbounded) operators. In contrast to this, in the following we will strictly adhere to the (bounded) context of a W^*-algebra M and positive linear forms.

To start discussions concerning Corollary 2(3)–(4), for given $\nu, \varrho \in M_+^*$, let us consider two real-valued functions on M, $\tau_{\nu,\varrho}$ and $\upsilon_{\nu,\varrho}$, which are defined at $z \in M$ by

$$\tau_{\nu,\varrho}(z) \;=\; \inf_{\{y,x\}\subset M,\, z=\sum_{j\leqslant n} y_j^* x_j} \frac{1}{2}\sum_j \left\{\nu\big(y_j^* y_j\big) + \varrho\big(x_j^* x_j\big)\right\}, \tag{2.3a}$$

$$\upsilon_{\nu,\varrho}(z) \;=\; \inf_{z=y^*x} \frac{1}{2}\left\{\nu(y^*y) + \varrho(x^*x)\right\}. \tag{2.3b}$$

Thereby, within the former expression, the infimum is to be taken over all finite double systems $\{y, x\}$ of operators of M obeying $z = \sum_{j\leqslant n} y_j^* x_j$, with $n \in \mathbb{N}$ arbitrarily chosen. For notational simplicity, we subsequently use the shortcut notation $z = \{y, x\}$ whenever such a type of relation occurs. If we want to consider only minimal systems of that kind ($n = 1$), which, e.g., is referred to in (2.3b), the condition $z = y^*x$ will be explicitly used.

Note that the assertions of Corollary 2(3)–(4) then read

$$\upsilon_{\nu,\varrho}(\mathbf{1}) = \tau_{\nu,\varrho}(\mathbf{1}) = \sqrt{P_M(\nu, \varrho)}\,. \tag{2.3c}$$

Also, it is obvious from the structure of the expression within definition (2.3a) that $\tau_{\nu,\varrho}$ is a seminorm, whereas from (2.3b) it is obvious that $\tau_{\nu,\varrho}$ is a lower bound for $\upsilon_{\nu,\varrho}$:

$$\tau_{\nu,\varrho}(z) \leqslant \upsilon_{\nu,\varrho}(z)\,. \tag{2.3d}$$

Remark that, in relativistic quantum field theory, it was to be hoped that seminorms of τ-type would be useful in proving the existence of non-trivial (weak) intertwiners between so-called standard representations [10, 29]. These standard representations roughly correspond to the cyclic $*$-representations of ν and ϱ in our bounded context (for the context, see also [18], especially Definition 2.2.14). Clearly, within specific settings this is the (highly nontrivial) analog over unbounded observable algebras of the (comparably trivial) task of analyzing the structure of the set $\Gamma_M(\nu, \varrho)$

in the bounded case. In case, the above idea reduces to enquiring about the upper bounds of $f \in \Gamma_M(\nu, \varrho)$ which can be read in terms of the seminorm $\tau_{\nu,\varrho}$, that is, one is looking for estimates by $\tau_{\nu,\varrho}$ in the form:

$$\forall\, z \in M \ : \ \ |f(z)| \leqslant c\, \tau_{\nu,\varrho}(z), \tag{2.3e}$$

for some real constant $c > 0$, for instance.

More precisely, the information concerning Corollary 2(3)–(4) consists of the following:

– the estimate (2.3e) holds with respect to the seminorm (2.3a), with $c = 1$, and this estimate, being the best possible in favor of the above task, that is, $\Gamma_M(\nu, \varrho)$ appears to be trivial, $\Gamma_M(\nu, \varrho) = \{0\}$, if and only if $\tau_{\nu,\varrho}$ is trivial, $\tau_{\nu,\varrho} \equiv 0$;
– the seminorm $\tau_{\nu,\varrho}$ can be calculated exactly, even if $\{y, x\}$, under the infimum in (2.3a), is bent to be varied only through minimal double systems with $z = y^*x$, i.e., according to this and (2.3b), one has $\tau_{\nu,\varrho} = \upsilon_{\nu,\varrho}$ to hold;
– when seen in the form of (2.3c), in generalizing from Corollary 2(3) for each $\nu, \varrho \in M_+^*$ and, given $z \in M$, an (heuristic useful) interpretation of the values of the seminorm $\tau_{\nu,\varrho}$ in terms of ‘transition probability’ (and, thus, in terms of the Bures distance) between certain inners derived from $\{\nu, \varrho\}$ positive linear forms can be given.

It is plain to see that the answers to the corresponding items can be read off as straightforward consequences of the following result:

COROLLARY 3. *For each $a, b \in M$ and $z = a^*b$, the following holds:*

$$\begin{aligned}\tau_{\nu,\varrho}(z) = \upsilon_{\nu,\varrho}(z) \ &= \sup_{f \in \Gamma_M(\nu,\varrho)} |f(z)| \\ &= \sqrt{P_M\big(\nu, \varrho^z\big)} = \sqrt{P_M\big(\nu^a, \varrho^b\big)}\,.\end{aligned} \tag{2.3f}$$

Proof. First note that each finite double system $\{y, x\}$ obeying $\mathbf{1} = \{y, x\}$ through setting $\tilde{y}_j = y_j a$ and $\tilde{x}_j = x_j b$, respectively, provides another finite double system of the same length $\{\tilde{y}, \tilde{x}\}$ with $a^*b = \{\tilde{y}, \tilde{x}\}$ (especially, minimal double systems will be transformed into minimal ones again). Hence, in view of Corollary 2(3)–(4) and (2.3a)–(2.3b) one can conclude as follows:

$$\begin{aligned}\sqrt{P_M\big(\nu^a, \varrho^b\big)} \ &= \ (1/2) \inf_{\mathbf{1}=\{y,x\}} \sum_j \big\{\nu^a\big(y_j^* y_j\big) + \varrho^b\big(x_j^* x_j\big)\big\} \\ &= \ (1/2) \inf_{\mathbf{1}=\{y,x\}} \sum_j \big\{\nu\big(\tilde{y}_j^* \tilde{y}_j\big) + \varrho\big(\tilde{x}_j^* \tilde{x}_j\big)\big\} \\ &\geqslant \ (1/2) \inf_{a^*b=\{y,x\}} \sum_j \big\{\nu\big(y_j^* y_j\big) + \varrho\big(x_j^* x_j\big)\big\} \\ &= \ \tau_{\nu,\varrho}(a^*b)\,.\end{aligned}$$

Thus, the following estimate has been established:

$$\tau_{\nu,\varrho}(a^*b) \leqslant \sqrt{P_M\big(\nu^a, \varrho^b\big)}\,. \tag{$\circ$}$$

Also, if from the pair $\{\nu, \varrho\}$ a representation $\{\pi, \mathcal{K}\}$ as in the premises of Theorem 2(3) is chosen, with fixed $\varphi \in \mathcal{S}_{\pi,M}(\nu)$ and $\psi \in \mathcal{S}_{\pi,M}(\varrho)$, then obviously also $\pi(a)\varphi \in \mathcal{S}_{\pi,M}(\nu^a)$ and $\pi(b)\psi \in \mathcal{S}_{\pi,M}(\varrho^b)$ are fulfilled. Application of (1.6) with respect to $\{\nu, \varrho\}$, $\{\nu^a, \varrho^b\}$ and $\{\nu, \varrho^z\}$ will yield that $\langle\pi(\cdot)k\psi, \varphi\rangle$, $\langle\pi(\cdot)k\pi(b)\psi, \pi(a)\varphi\rangle$ and $\langle\pi(\cdot)k\pi(z)\psi, \varphi\rangle$, respectively, will be running through all of $\Gamma_M(\nu, \varrho)$, $\Gamma_M(\nu^a, \varrho^b)$ and $\Gamma_M(\nu, \varrho^z)$, respectively, if k is supposed to be varied through all of $(\pi(M)')_1$. Now, for each $k \in (\pi(M)')_1$, one has $\langle k\pi(b)\psi, \pi(a)\varphi\rangle = \langle k\pi(z)\psi, \varphi\rangle = \langle\pi(z)k\psi, \varphi\rangle$. Hence, in line with Theorem 2(2), when the latter is accordingly applied to these three special situations, under the premise of $z = a^*b$ the estimate ($\circ$) can be continued as follows:

$$\tau_{\nu,\varrho}(z) \leqslant \sqrt{P_M\big(\nu^a, \varrho^b\big)} = \sqrt{P_M\big(\nu, \varrho^z\big)} = \sup_{f\in\Gamma_M(\nu,\varrho)} |f(z)|. \tag{$\circ'$}$$

Now, suppose that $z = \{y, x\}$ within the context of $\{\nu, \varrho\}$. By definition of $\Gamma_M(\nu, \varrho)$, for $f \in \Gamma_M(\nu, \varrho)$, one has

$$\begin{aligned} |f(z)| &\leqslant \sum_j |f\big(y_j^* x_j\big)| \leqslant \sum_j \sqrt{\nu\big(y_j^* y_j\big)\,\varrho\big(x_j^* x_j\big)} \\ &\leqslant \frac{1}{2}\sum_j \big\{\nu\big(y_j^* y_j\big) + \varrho\big(x_j^* x_j\big)\big\}. \end{aligned}$$

From this and in view of (2.3a), $\sup_{f\in\Gamma_M(\nu,\varrho)}|f(z)| \leqslant \tau_{\nu,\varrho}(z)$ follows, which with the help of (2.3d) can be turned into

$$\sup_{f\in\Gamma_M(\nu,\varrho)} |f(z)| \leqslant \tau_{\nu,\varrho}(z) \leqslant \upsilon_{\nu,\varrho}(z). \tag{$\circ''$}$$

On the other hand, for $\varepsilon > 0$, Corollary 2(4) can be applied to the pair $\{\nu, \varrho^z\}$ and yields invertible $a > \mathbf{0}$ obeying

$$\sqrt{P_M(\nu, \varrho^z)} + \varepsilon \geqslant (1/2)\,\big\{\nu(a) + \varrho^z(a^{-1})\big\}.$$

Let us define $y = \sqrt{a}$ and $x = \sqrt{a}^{\,-1} z$. Then, $z = y^*x$ and $\big\{\nu(a) + \varrho^z(a^{-1})\big\} = \big\{\nu(y^*y) + \varrho(x^*x)\big\}$ are fulfilled. Hence, in view of the above, $\upsilon_{\nu,\varrho}(z) \leqslant \sqrt{P_M(\nu, \varrho^z)}$ $+\varepsilon$ can be followed. Since $\varepsilon > 0$ can be taken at will from the latter in accordance with (2.3b), we get $\upsilon_{\nu,\varrho}(z) \leqslant \sqrt{P_M(\nu, \varrho^z)}$. Upon taking this together with ($\circ''$) and ($\circ'$), we can conclude that in fact equality has to occur within ($\circ''$) and ($\circ'$), i.e., (2.3f) holds. This closes the proof of Corollary 3.

Proof of Theorem 3. The formula of Theorem 3(1) is given by one of the particular subequations coming along with (2.3f). Moreover, according to another subequation of (2.3f), $\upsilon_{\nu,\varrho}(z) = P_M(\nu^a, \varrho^b)^{1/2}$ holds. Inserting this into (1.3), in view of (2.3b), yields

$$\begin{aligned} d_{\mathsf{B}}(\nu^a, \varrho^b)^2 &= \nu(a^*a) + \varrho(b^*b) - \inf_{z=y^*x}\bigl\{\nu(y^*y) + \varrho(x^*x)\bigr\} \\ &= \sup_{z=y^*x}\bigl\{\nu(a^*a - y^*y) + \varrho(b^*b - x^*x)\bigr\}, \end{aligned}$$

which is Theorem 3(2).

Remark 2. (1) Without proof, we remark that $P_M(\nu, \varrho) = 0$ is equivalent to $\nu \perp \varrho$ (see, e.g., [5]). Recall that the orthogonality of two C^*-algebraic positive linear forms ν, ϱ is defined as $\|\nu - \varrho\|_1 = \|\nu\|_1 + \|\varrho\|_1$.

(2) Especially for states ν, ϱ occurring along with quantum physical problems over an algebra of observables M, one is inclined to give $P_M(\nu, \varrho)$ a (quantum) probabilistic interpretation. Corollary 3 within such a context will tell us that an interpretation which reads in terms of the transition probability, but now between the 'perturbed' states ν^a and ϱ^b, also extends to the value of the rather abstractly defined seminorms $M \ni z \longmapsto \tau_{\nu,\varrho}(z)$ at $z = a^*b$. Thus, if to given pair $\{\nu, \varrho\}$ of states and in accordance with (2.3f) and the previous item (1), those operators a, b are considered which are solutions of the equation $\tau_{\nu,\varrho}(a^*b) = 0$ (and for which both ν^a and ϱ^b are states again), then these might be interpreted as all possible elementary 'operations' (i.e. inner implementable perturbations) driving $\{\nu, \varrho\}$ into mutually orthogonal states.

(3) Due to the mentioned interpretation of the values of the seminorm $\tau_{\nu,\varrho}$ in terms of $\sqrt{P_M}$, which manifests itself by (2.3f), some subadditivity property of $\sqrt{P_M}$ in respect to inner derived positive linear forms can be followed:

$$a^*b = \sum_{j\leqslant n} a_j^* b_j \implies \sqrt{P_M\bigl(\nu^a, \varrho^b\bigr)} \leqslant \sum_{j\leqslant n}\sqrt{P_M\bigl(\nu^{a_j}, \varrho^{b_j}\bigr)}\,.$$

(4) The fact that $\tau_{\nu,\varrho} = \upsilon_{\nu,\varrho}$ holds is mainly due to our restriction to *bounded* operator algebras and cannot be expected to extend simply to a context with $*$-algebras of unbounded operators.

2.2.3. *Comments on Corollary 2(5)*: *Minimizing Abelian Algebras*

That Corollary 2(5) is a notable result on its own rights – and is not something to be easily abandoned – has been recognized only recently, and as such will be discussed here (and in more detail in the next section) for the first time.

In comparing the item in question with Theorem 2(1), one immediately notices that the essential difference with the latter result lies in the fact that under the infimum instead of a geometrical means, the arithmetical means of the same two expressions now enters the equation.

Quite naturally, within the context of Corollary 2(5) (and within the context of Theorem 2(1) as well), a main interest will be in describing the structure of those invertible $x \in M_+$ from which, by the expression of $\frac{1}{2}\left\{\nu(x) + \varrho(x^{-1})\right\}$ (or $\sqrt{\nu(x)\,\varrho(x^{-1})}$, respectively), the (common) infimum $\sqrt{P_M(\nu,\varrho)}$ is nearly attained. Such problems and related questions will now be discussed. As such, for the purposes of estimation theory, Corollary 2(5) seems to be better suited than Theorem 2(1). For instance, the map $x \longmapsto \frac{1}{2}\left\{\nu(x) + \varrho(x^{-1})\right\}$ is more sensitive to certain variations of the positive invertible operator $x \in M$ than the map $x \longmapsto \sqrt{\nu(x)\,\varrho(x^{-1})}$ is (compare the behavior of both under the change $x \mapsto \lambda\, x$, for real $\lambda > 0$, simply).

Relating the quality of the mentioned approximation, one has the following simple facts (cf. also Theorem 4.4 in [2]).

COROLLARY 4. *Let $\nu, \varrho \in M_+^*$, and be $\{x\} \subset M_+$ a sequence of invertible elements. The following facts are equivalent*:

(1) $\sqrt{P_M(\nu,\varrho)} = \lim_{n\to\infty} \frac{1}{2}\{\nu(x_n) + \varrho(x_n^{-1})\}$;
(2) $\sqrt{P_M(\nu,\varrho)} = \lim_{n\to\infty} \nu(x_n) = \lim_{n\to\infty} \varrho(x_n^{-1})$.

Moreover, if Comm$[M]$ *is the family of all Abelian* W*-*subalgebras of M with the same unity as M, then one has*

(3) $P_M(\nu,\varrho) = \inf_{R\in \mathrm{Comm}[M]} P_R(\nu|_R, \varrho|_R)$.

Proof. In view of Equations (2.1), the asserted equivalence immediately follows from Theorem 2(1) and Corollary 2(5). Also (3) can be seen as an obvious consequence of each of these items.

Now, for a given pair $\{\nu,\varrho\}$ of positive linear forms, a set $\mathsf{Min}_M(\nu,\varrho)$ will be defined as

$$\mathsf{Min}_M(\nu,\varrho) = \left\{ x \in M_+ :\ \sqrt{P_M(\nu,\varrho)} = \frac{1}{2}\left\{\nu(x) + \varrho(x^{-1})\right\}\right\}.$$

The elements of $\mathsf{Min}_M(\nu,\varrho)$ will be called *minimizing* (positive invertible) elements of the pair $\{\nu,\varrho\}$, where, in this notation Corollary 2(5) is tacitly referred to within context.

Note that since the set of all invertible positive elements is neither compact nor closed, it is a nontrivial problem to decide from a concrete pair $\{\nu,\varrho\}$ of positive linear forms whether or not the infimum within Corollary 2(5) is a minimum.

In fact, general this cannot happen, as the following simple counterexample shows.

EXAMPLE 3. According to elementary spectral theory for invertible $y \in M_+$, one has $y \geqslant \|y^{-1}\|^{-1}\mathbf{1}$. Hence, for each pair $\{\nu,\varrho\} \neq \{0,0\}$ of positive linear forms and for each invertible $x \in M_+$ one infers that

$$\{\nu(x) + \varrho(x^{-1})\}/2 \geqslant \{\|\nu\|_1/\|x^{-1}\| + \|\varrho\|_1/\|x\|\}/2 > 0$$

has to be fulfilled. On the other hand, according to Remark 2(2), in the special case of $\nu \perp \varrho$, one has $\sqrt{P_M(\nu,\varrho)} = 0$. Thus, in view of the previous estimate in the case of a nontrivial pair of mutually orthogonal positive linear forms, $\mathsf{Min}_M(\nu,\varrho) = \emptyset$ holds.

On the other hand, there also exist classes where this question can be answered affirmatively. A criterion relating to this matter is easily obtained from Corollary 4(1)–(2) and reads as follows:

$$x \in M_+,\ \sqrt{P_M(\nu,\varrho)} = \nu(x) = \varrho\big(x^{-1}\big) \iff x \in \mathsf{Min}_M(\nu,\varrho)\,. \tag{2.4}$$

EXAMPLE 4. Suppose $\varrho = \nu^a$, with $a \in M_+$ being *invertible*. Then, in view of Theorem 1, the criterion (2.4) becomes applicable with $x = a$ and shows that the infimum in Corollary 2(5) is a minimum.

Let us refer to an Abelian W^*-subalgebra $R \subset M$ with $\mathbf{1} \in R$ as the *minimizing Abelian subalgebra* if the infimum within Corollary 4(3) is a minimum and is attained at R. For instance, if $\mathsf{Min}_M(\nu,\varrho) \neq \emptyset$ is fulfilled, then in line with the above, the infimum is attained at each subalgebra R which is generated by $\mathbf{1}$ and some particular $x \in \mathsf{Min}_M(\nu,\varrho)$. Thus, in generalizing the problem on the existence of minimizing elements, a more general question on the existence of minimizing Abelian subalgebras naturally arises.

3. Special Subjects

3.1. MINIMIZING ELEMENTS

In this section we inquire about the existence and uniqueness of minimizing positive invertible elements, and we derive some results on the structure of $\mathsf{Min}_M(\nu,\varrho)$. Let $x, z \in M_+$ be any two invertible positive elements. Let $\delta = (z - x)$. Then the following algebraic identity can be easily checked to hold:

$$z^{-1} = x^{-1} - x^{-1}\delta x^{-1} + \Delta(z,x)\,, \tag{3.1a}$$

where $\Delta(z,x) = m(z,x)^* m(z,x)$ holds, and $m(z,x)$ is defined by

$$m(z,x) = (x^{-1/2}\delta x^{-1/2})(x^{-1/2} z x^{-1/2})^{-1/2} x^{-1/2}\,. \tag{$\star$}$$

By construction of $\Delta(z,x)$ and by invertibility of z, x from ($\star$), the following can be followed

$$\Delta(z,x) \in M_+, \text{ with } \big\{\Delta(z,x) = \mathbf{0} \iff \delta = \mathbf{0}\big\}. \tag{3.1b}$$

Also, since $x^{-1/2}\delta x^{-1/2}$ is commuting with $x^{-1/2} z x^{-1/2}$, yet another expression for $m(z,x)$ can be obtained from ($\star$). This reads as

$$m(z,x) = (x^{-1/2} z x^{-1/2})^{-1/2} x^{-1/2}\delta x^{-1}\,. \tag{3.1c}$$

With the help of (3.1a) and the previous notations, one finds

$$\begin{aligned}&\frac{1}{2}\big\{\nu(z)+\varrho(z^{-1})\big\}-\frac{1}{2}\big\{\nu(x)+\varrho(x^{-1})\big\}\\&\quad=\frac{1}{2}\big\{\nu(\delta)-\varrho(x^{-1}\delta x^{-1})\big\}+\frac{1}{2}\varrho(\Delta(z,x))\,.\end{aligned}\tag{3.1d}$$

Note that the set M_+^{inv} of all invertible positive elements of M is an open, nonpointed subcone within the real Banach space $\{M_{\mathrm{h}}, \|\cdot\|\}$ of the Hermitian portion of M. Hence, for a particular $x\in M_+^{\mathsf{inv}}$ and given $y\in M_{\mathrm{h}}$, for all $t\in\mathbb{R}$ sufficiently small, $z_t=x+ty\in M_+^{\mathsf{inv}}$ has to hold (one might take $|t|<\|x^{-1}yx^{-1}\|^{-1}$). In this special situation formula (3.1d) at such a parameter t reads as

$$\begin{aligned}&\frac{1}{2}\big\{\nu(z_t)+\varrho(z_t^{-1})\big\}-\frac{1}{2}\big\{\nu(x)+\varrho(x^{-1})\big\}\\&\quad=\frac{t}{2}\big\{\nu(y)-\varrho(x^{-1}yx^{-1})\big\}+\frac{t^2}{2}\varrho(\Delta_t(y|x))\,,\end{aligned}\tag{3.2}$$

where $\Delta_t(y|x)=t^{-2}\Delta(z_t,x)$ is defined for $t\neq 0$ and, at $t=0$, we let

$$\Delta_0(y|x)=\|\cdot\|-\lim_{t\to 0}t^{-2}\Delta(z_t,x)=x^{-1}yx^{-1}yx^{-1}.$$

We are now ready for the following redefinition of $\mathsf{Min}_M(\nu,\varrho)$.

PROPOSITION 2. *For any $\nu,\varrho\in M_+^*$ the following holds*:

$$\mathsf{Min}_M(\nu,\varrho)=\big\{x\in M_+^{\mathsf{inv}}:\ \nu(y)=\varrho(x^{-1}yx^{-1}),\ \forall\, y\in M_{\mathrm{h}}\big\}\,.\tag{3.3}$$

Proof. Suppose $x\in\mathsf{Min}_M(\nu,\varrho)$. Then, for each fixed $y\in M_{\mathrm{h}}$ and for all $t\in\mathbb{R}\backslash\{0\}$ sufficiently small, in accordance with (3.2)

$$-\big|\nu(y)-\varrho(x^{-1}yx^{-1})\big|\geqslant -|t|\,\varrho(\Delta_t(y|x))$$

has to hold. Having in mind that according to the above, $t\mapsto\Delta_t(y|x)$ is norm-continuous at $t=0$, one then has $\lim_{t\to 0}|t|\,\varrho(\Delta_t(y|x))=0$. In view of the previous estimate, $\nu(y)=\varrho(x^{-1}yx^{-1})$ is obtained.

On the other hand, assume that $x\in M_+^{\mathsf{inv}}$ such that, for each $y\in M_{\mathrm{h}}$, $\nu(y)=\varrho(x^{-1}yx^{-1})$ is satisfied. For each other $z\in M_+^{\mathsf{inv}}$, let $\delta=(z-x)=y$. Then one especially has $\{\nu(\delta)-\varrho(x^{-1}\delta x^{-1})\}=0$. Hence, (3.1d) can be applied and, owing to the positivity of $\Delta(z,x)$ and ϱ, yields $\frac{1}{2}\big\{\nu(z)+\varrho(z^{-1})\big\}-\frac{1}{2}\big\{\nu(x)+\varrho(x^{-1})\big\}\geqslant 0$. Hence, since z can be arbitrarily chosen from M_+^{inv}, $x\in\mathsf{Min}_M(\nu,\varrho)$ follows. This completes the proof of (3.3).

After these preliminaries, we may now summarize as follows.

THEOREM 4. *Let M be a W^*-algebra. For $\nu,\varrho\in M_+^*$ one has*

(1) $\mathsf{Min}_M(\nu,\varrho)\neq\emptyset\iff\exists\, a\in M_+^{\mathsf{inv}}:\ \varrho=\nu^a$;

(2) $\mathsf{Min}_M(\nu, \varrho) = \{x + I_\nu\} \cap M_+^{\mathsf{inv}}, \ \forall\, x \in \mathsf{Min}_M(\nu, \varrho)$;
(3) $\#\mathsf{Min}_M(\nu, \varrho) = 1 \iff \exists\, a \in M_+^{\mathsf{inv}} : \varrho = \nu^a, \ \nu$ *is faithful*.

Proof. According to Example 4, for $\varrho = \nu^a$ with $a \in M_+^{\mathsf{inv}}$, one has $a \in \mathsf{Min}_M(\nu, \varrho)$. On the other hand, if $\mathsf{Min}_M(\nu, \varrho) \neq \emptyset$ is supposed in line with formula (3.3) and since linear forms on a C^*-algebra are uniquely determined through their values for the Hermitian portion, $\nu = \varrho(x^{-1}(\cdot)x^{-1})$ has to be fulfilled for some $x \in M_+^{\mathsf{inv}}$. That is, $\varrho = \nu^a$ holds with $a = x$. In summarizing, (1) is valid.

To see (2), suppose that $x \in \mathsf{Min}_M(\nu, \varrho)$ and be $z \in M_+^{\mathsf{inv}}$. According to what has been said previously $\varrho = \nu^x$ and, therefore, from (3.3) and (3.1d) one infers that $z \in \mathsf{Min}_M(\nu, \varrho)$ happens if and only if $\nu(x\Delta(z, x)x) = 0$ is fulfilled. By construction of $\Delta(z, x)$, the latter is equivalent with $m(z, x)x \in I_\nu$, see (1.1b). According to (3.1c), the latter is the same as $(x^{-1/2}zx^{-1/2})^{-1/2}x^{-1/2}\delta \in I_\nu$, with $\delta = (z - x)$. Since I_ν is a left ideal and $(x^{-1/2}zx^{-1/2})^{-1/2}x^{-1/2}$ is invertible, from this we finally conclude that, for $z \in M_+^{\mathsf{inv}}$, the condition $z \in \mathsf{Min}_M(\nu, \varrho)$ has to be equivalent with $\delta \in I_\nu$. Owing to $\mathsf{Min}_M(\nu, \varrho) \subset M_+^{\mathsf{inv}}$ this is (2).

In order to see (3), we first remark that for faithful ν one has $I_\nu = \{\mathbf{0}\}$. Hence, from the just proved (2), the uniqueness of a minimizing element evidently follows. On the other hand, for an eventually existing $r \in I_\nu \backslash \{\mathbf{0}\}$, owing to $r^*r = |r|^2$, also $|r| \in I_\nu \backslash \{\mathbf{0}\}$ follows, see (1.1b). Hence, since $a \in \mathsf{Min}_M(\nu, \varrho)$ is invertible, by standard facts and owing to $z \geqslant a$, also $z \in M_+^{\mathsf{inv}}$ follows for $z = a + |r|$. By (2) this, however, then implies that $z \in \mathsf{Min}_M(\nu, \varrho)$. Since $z \neq a$ holds, we therefore have $\#\mathsf{Min}_M(\nu, \varrho) > 1$, for nonfaithful ν. Taking this together with what has been said previously yields (3).

Since $\mathsf{Min}_M(\varrho, \nu) = \{x^{-1} : x \in \mathsf{Min}_M(\nu, \varrho)\}$ holds, from Theorem 4(2), for $\mathsf{Min}_M(\nu, \varrho) \neq \emptyset$, one infers that both positive linear forms have to be faithful or not, only simultaneously. By reversing this, another class of counterexamples is easily obtained.

EXAMPLE 5. Let $\nu, \varrho \in M_+^*$. Suppose that one of the two forms is faithful. Then, the infimum in Corollary 2(5) cannot be attained on the invertible positive elements of M.

Remark 3. According to Theorem 4(1), minimizing elements can exist if and only if each of the two positive linear forms of a pair $\{\nu, \varrho\}$ can be inner-derived by means of some positive invertible element from the other one. All these cases are covered by Example 4.

As announced at the end of Section 2.2.3, the next best question to be raised concerns the existence of a commutative W^*-subalgebra R of M, with $\mathbf{1} \in R$, such that the infimum in Corollary 4(3) could be attained.

3.2. MINIMIZING COMMUTATIVE SUBALGEBRAS

We start with examples where minimizing abelian subalgebras exist but which are found to be slighly beyond the bounds of Example 4.

EXAMPLE 6. Suppose $\varrho = \nu^a$, for $a \in M_+$. By functional calculus (use the spectral representation theorem within the W*-algebra M) one infers that $a(a + \varepsilon\,\mathbf{1})^{-1}a \leqslant a$ holds, for each real $\varepsilon > 0$. Hence, $a_\varepsilon = (a + \varepsilon\,\mathbf{1}) \in M_+^{\mathrm{inv}}$ with $\varrho(a_\varepsilon^{-1}) \leqslant \nu(a)$. Owing to this, Theorems 2(1) and 1 (or, equivalently (1.7)),

$$\nu(a) = \sqrt{P_M(\nu, \varrho)} \leqslant \nu(a_\varepsilon) = \nu(a) + \varepsilon\,\|\nu\|_1$$

as well as

$$\nu(a)^2 = P_M(\nu, \varrho) \leqslant \varrho(a_\varepsilon^{-1})\,\nu(a_\varepsilon) \leqslant \nu(a)^2 + \varepsilon\,\nu(a)\|\nu\|_1$$

are obtained. Upon performing the limit $\varepsilon \to 0$ in both relations and regarding Corollary 4(1)–(2) will give that the W*-subalgebra generated by a and $\mathbf{1}$ can be chosen as minimizing commutative subalgebra R.

The fact that a subalgebra R can be minimizing for a given pair $\{\nu, \varrho\}$ implies that some very specific additional conditions have to be fulfilled. An important instance of such conditions occurs within the context of those minimizing subalgebras which come from Example 6.

LEMMA 1. *Suppose $\nu, \varrho \in M_+^*$ and let R be a W*-subalgebra of M such that $\varrho|_R = (\nu|_R)^a$ holds for some $a \in R_+$. Then, whenever R is minimizing for $\{\nu, \varrho\}$, the relation*

$$\nu^p(a) - \nu^{p^\perp}(a) = \nu(a) \tag{3.4}$$

holds for each orthoprojection $p \in M$ obeying $p^\perp \in I_\varrho$.

Proof. Let $P = P_M(\nu, \varrho)$. The assumption that R can be minimizing together with the reasoning of Example 6 when applied in respect of $\{\nu|_R, \varrho|_R\}$ over R, prove that for $a_\varepsilon = a + \varepsilon\,\mathbf{1}$ with $\varepsilon > 0$, one has

$$\sqrt{P} = \nu(a) = \lim_{\varepsilon\to 0} \nu(a_\varepsilon) = \lim_{\varepsilon\to 0} \varrho(a_\varepsilon^{-1}).$$

Now, let $u = p + \lambda\, p^\perp$, with real $\lambda \neq 0$. Define $a_\varepsilon(\lambda) = u^* a_\varepsilon u$. Then, for each $\varepsilon > 0$, one has $a_\varepsilon(\lambda) \in M_+^{\mathrm{inv}}$. Note also that the assumption on p saying that $p^\perp \in I_\varrho$ is fulfilled together with the special structure of u imply

$$\varrho(y) = \varrho(pyp) = \varrho(u^* y u) = \varrho(u^{-1} y u^{-1*})$$

to be fulfilled for each $y \in M$. Hence, by construction of $a_\varepsilon(\lambda)$, $\lim_{\varepsilon\to 0} \varrho(a_\varepsilon(\lambda)^{-1}) = \lim_{\varepsilon\to 0} \varrho(a_\varepsilon^{-1}) = \sqrt{P}$ especially follows. On the other hand, since

$$\nu(a_\varepsilon(\lambda)) = \nu^p(a_\varepsilon) + 2\lambda\,\Re\,\nu(p^\perp a_\varepsilon p) + \lambda^2 \nu^{p^\perp}(a_\varepsilon)$$

is fulfilled, in view of the above, one arrives at

$$\lim_{\varepsilon\to 0} \nu(a_\varepsilon(\lambda)) = \nu^p(a) + 2\lambda\,\Re\,\nu(p^\perp ap) + \lambda^2 \nu^{p^\perp}(a).$$

Note that, according to Theorem 2(1), the estimate $\lim_{\varepsilon\to 0} \nu(a_\varepsilon(\lambda))\,\varrho(a_\varepsilon(\lambda)^{-1}) \geqslant P$ has to be fulfilled, which in view of the previous, amounts to requiring

$$\begin{aligned}&\sqrt{P}\,\{\nu^p(a) + 2\lambda\,\Re\,\nu(p^\perp ap) + \lambda^2\nu^{p^\perp}(a)\}\\ &\quad\geqslant\; P = \sqrt{P}\,\{\nu^p(a) + 2\,\Re\{\nu(p^\perp ap)\} + \nu^{p^\perp}(a)\},\end{aligned}$$

for all reals $\lambda \neq 0$. That is,

$$2\sqrt{P}(\lambda - 1)\left\{\Re\,\nu(p^\perp ap) + \frac{1}{2}(\lambda + 1)\,\nu^{p^\perp}(a)\right\} \geqslant 0$$

has to be fulfilled, for each real $\lambda \neq 0$.

Suppose $P \neq 0$ first. In considering the previous estimate for $\lambda > 1$, one infers that $\Re\,\nu(p^\perp ap) + \frac{1}{2}(\lambda + 1)\,\nu^{p^\perp}(a) \geqslant 0$ has to be fulfilled, whereas for $\lambda < 1$ we see that $\Re\,\nu(p^\perp ap) + \frac{1}{2}(\lambda + 1)\,\nu^{p^\perp}(a) \leqslant 0$ has to be fulfilled. Upon performing the limits $\lambda \searrow 1$ and $\lambda \nearrow 1$ within the mentioned relations for $\lambda > 1$ and $\lambda < 1$, respectively, and then comparing the results will show that $\nu^{p^\perp}(a) = -\Re\,\nu(p^\perp ap)$ has to be fulfilled. By means of this,

$$\nu(a) = \nu^p(a) + 2\,\Re\,\nu(p^\perp ap) + \nu^{p^\perp}(a) = \nu^p(a) - \nu^{p^\perp}(a)$$

is seen. This proves the result in case of $P \neq 0$.

Finally, for $P = 0$, one has $\nu(a) = 0$. Owing to $a \geqslant \mathbf{0}$, $a \in I_\nu$. Hence, also $0 = \nu(pa) = \nu(ap)$ and therefore from

$$\nu^{p^\perp}(a) = \nu(a) - 2\,\Re\,\nu(ap) + \nu^p(a),$$

one gets $\nu^p(a) - \nu^{p^\perp}(a) = 0$ which is in accordance with (3.4) in this special case.

Bearing Example 5 in mind, we remark that, for faithful ν and $\varrho = \nu^a$, with $a \in M_+$ and $\ker a \neq \{\mathbf{0}\}$, the most simple situations arise where Example 6 provides cases which go beyond the bounds of Example 4. Less trivial situations of that kind arise from generalizing Example 2 and modifying those arguments, along the lines of which we have been following within Example 6. The result in question, which will be proved here in a sketchy way, reads as follows:

PROPOSITION 3. *Let $\{\nu, \varrho\}$, with normal $\nu, \varrho \in M_+^*$, and support orthoprojections which are mutually $\leqslant$-comparable, say $s(\varrho) \leqslant s(\nu)$, be fulfilled. Then a minimizing commutative* W*-*subalgebra R of M exists.*

Sketch of proof. We remark first that for *normal* positive linear forms ν, ϱ with supports obeying $s(\varrho) \leqslant s(\nu)$, the problem in question by way of an appropriately chosen *normal* *-representation $\{\pi, \mathcal{K}\}$ which obeys $\mathcal{S}_{\pi,M}(\nu) \neq \emptyset$ and

$\mathcal{S}_{\pi,M}(\varrho) \neq \emptyset$, can always be reduced to the analogous problem over the vN-algebra $N = \pi(M)''$. In this setting, with given $\varphi \in \mathcal{S}_{\pi,M}(\nu)$, the assumption about the supports can be shown to ensure the existence of some (possibly unbounded) selfadjoint positive linear operator A which is affiliated to N and which obeys $\psi = A\varphi \in \mathcal{S}_{\pi,M}(\varrho)$. Note that since A is affiliated to N, the operator A can be chosen to be independent of the particularly chosen φ within $\mathcal{S}_{\pi,M}(\nu)$. Let ν_π and ϱ_π be the vector functionals generated by φ and ψ over the vN-algebra N. Extending the notion 'inner derived positive linear form' slighly to include at least such situations with vector forms on N and (unbounded) positive selfadjoint linear operators affiliated with N, for $\varrho_\pi = \nu_\pi^A$, one easily proves that formula (1.7) remains true in the sense of $\sqrt{P_N(\nu_\pi, \varrho_\pi)} = \nu_\pi(A) = \langle A\varphi, \varphi\rangle$. Since then also the arguments raised within the context of Example 6 are easily justified to remain valid with $A_\varepsilon = A + \varepsilon\,\mathbf{1}$ instead of a_ε, following along the same line of conclusions as in Example 6, will provide $P_N(\nu_\pi, \varrho_\pi) = P_R(\nu_\pi|_R, \varrho_\pi|_R)$, with R being the commutative vN-subalgebra of N generated by the spectral resolution of A. Finally, since $P_N(\nu_\pi, \varrho_\pi) = P_M(\nu, \varrho)$ is always fulfilled (note that $\nu_\pi \circ \pi = \nu$ and $\varrho_\pi \circ \pi = \varrho$ hold), in view of the *normality* of π, which implies that even $N = \pi(M)$ holds, the just-mentioned result about ν_π, ϱ_π over N can be easily rewritten into one over M.

3.3. LEAST MINIMIZING COMMUTATIVE SUBALGEBRA

3.3.1. *Generalities on the Problem*

It is plain to see (from each of the items of Corollary 2, for instance) that the map $R \longmapsto \sqrt{P_R(\nu|_R, \varrho|_R)}$, $\nu, \varrho \in M_+^*$, with respect to the inclusion $\subset$ between W*-subalgebras of M behaves $\leqslant$-(anti-)monotoneous. Hence, if there is a minimizing commutative subalgebra R, then also each commutative subalgebra larger than this has to be minimizing.

Going the other way around within this context is less trivial. For instance, one might ask for the existence of a *least*-minimizing commutative W*-subalgebra of M with the same unit. In the case of the existence of a least-minimizing subalgebra, the latter will be denoted by $\mathcal{R}_M(\nu, \varrho)$.

Note that a least-minimizing subalgebra must not exist in either case of a pair $\{\nu, \varrho\}$ where a minimizing commutative subalgebra exists. To formulate a result for this, for the following, make use of $R[x]$ as the notation for the commutative W*-subalgebra of M which is generated by $\mathbf{1}$ and the Hermitian element $x \in M_{\mathrm{h}}$. Then, the simplest counterexamples against the existence of a least-minimizing algebra can be generated from the following auxiliary construction:

LEMMA 2. *Suppose $\varrho = \nu^x$ holds with $x \in M_+$. Then, for each $k \in I_\nu \cap M_+$, $R[x + k]$ is a minimizing Abelian subalgebra to $\{\nu, \varrho\}$. In the case where*

$$\varrho \notin \mathbb{R}_+\,\nu\,, \quad \bigcap_{k \in I_\nu \cap M_+} R[x + k] = \mathbb{C} \cdot \mathbf{1} \tag{3.5}$$

is fulfilled, there cannot exist a least-element among all minimizing commutative subalgebras to the pair $\{\nu, \varrho\}$.

Proof. It is easily inferred from (1.1a) and (1.1b) that $\varrho = \nu^{x+k}$ holds also for each $k \in I_\nu \cap M_+$. By Example 6, $R[x + k]$ will be a special minimizing commutative subalgebra. Also, from Definition 2 and with the help of known properties of the Cauchy–Schwarz inequality, one easily infers that, for each pair $\{\nu, \varrho\}$ of positive linear forms, $\sqrt{P_M(\nu, \varrho)} \leqslant \sqrt{\|\nu\|_1\|\varrho\|_1}$ is fulfilled, with equality occurring if and only if $\varrho = \lambda \cdot \nu$ happens for some nonnegative real λ. On the other hand, from the structure of Corollary 2(5) it is easily seen that $\sqrt{P_M(\nu, \varrho)} = \sqrt{\|\nu\|_1\|\varrho\|_1}$ is equivalent to the fact that $\mathbb{C} \cdot \mathbf{1}$ is among the minimizing subalgebras. Now, assume ν, ϱ as in (3.5). Then, according to the first of the previously-mentioned facts, the second condition in (3.5), in the case of the existence of a least-minimizing subalgebra, implied the latter to be trivial, whereas by the first condition in (3.5) and owing to the second of the above-mentioned facts, the trivial algebra $\mathbb{C} \cdot \mathbf{1}$ is excluded from being a minimizing subalgebra. Thus, a least-minimizing subalgebra cannot exist in this case.

Unfortunately, condition (3.5) can be easily satisfied, e.g. it can be shown to be fulfilled for any two noncommuting pure states (the following 2×2 case can exemplarily stand for any situation of this kind; we omit the details).

EXAMPLE 7. Let $M = \mathsf{M}_2(\mathbb{C})$ be the full algebra of 2×2-matrices with complex entries, $p, q \in M$ one-dimensional orthoprojections, with $[p, q] = pq - qp \neq \mathbf{0}$. Let $x = p + \varepsilon p^\perp$, with $0 < \varepsilon < 1$, and be $\nu \in M_+^* \backslash \{0\}$ with $\nu(q) = 0$ (such positive linear form trivially exists). Define $\varrho = \nu^x$. Then $q \in I_\nu \cap M_+$ and, in line with the first part of Lemma 2 for both x and $y = x + q$, one has that $R[x]$ and $R[y]$ are minimizing commutative subalgebras which, owing to the assumptions, obey $[x, y] \neq \mathbf{0}$ and, therefore, both have to be nontrivial as well as not being the same, $R[x] \neq R[y]$. Since each nontrivial commutative subalgebra of $\mathsf{M}_2(\mathbb{C})$ can be generated by exactly two atoms, then $R[x] \cap R[y] = \mathbb{C} \cdot \mathbf{1}$ has to be followed. This especially means that condition (3.5) is fulfilled and, thus, in accordance with the other assertion of Lemma 2, a least-minimizing subalgebra cannot exist.

The above negative result and the previous counter-example, together with some view on the structure of condition (3.5), indicate that the existence of a least-minimizing Abelian subalgebra seems to depend on the size as well as on the mutual position of the kernel ideals I_ν and I_ϱ in relation to each other (cf. also Lemma 1). Recall that the kernel ideal I_ν in a W^*-algebra becomes manageable, especially if ν is supposed to be *normal*. In this case, $I_\nu = Ms(\nu)^\perp$ holds, where $s(\nu)$ is the support orthoprojection of the normal positive linear form ν (be careful about the context; the same notation $s(x)$ will be also used for the support of a Hermitian element $x \in M_{\mathrm{h}}$ which will subsequently also play a rôle). Unfortunately, even in the normal case, only very few answers are known on this subject, except when we are in the special case with $\varrho \ll \nu$ which relates to Example 2, and where sufficiently many examples of minimizing Abelian subalgebras are known. Before going into details, some auxiliary notion relating to the general pair $\{\nu, \varrho\}$ of *normal* positive linear forms will be introduced:

DEFINITION 3. Let $R \subset M$ be a W*-subalgebra of M which contains the unity of M. R is called $\{\nu, \varrho\}$-*projective* provided the condition

$$\forall y \in R : \;\; \nu^{s(\varrho)}(y) = \nu(ys(\varrho)) \tag{3.6}$$

is fulfilled (R will be simply referred to as a *projective subalgebra* if the ordered pair is unambiguously given by the context).

EXAMPLE 8. For a normal, positive linear form ν, the unital subalgebra M^ν defined by

$$M^\nu = \{x \in M : \nu(xy) = \nu(yx),\ \forall\, y \in M\}$$

is a W*-subalgebra of M, which is usually called a ν-centralizer. Obviously, if the support $s(\varrho)$ of another normal positive linear form ϱ obeys $s(\varrho) \in M^\nu$, then relation (3.6) is automatically fulfilled for each W*-subalgebra R of M. Hence, in this case, each such R is $\{\nu, \varrho\}$-projective.

Remark 4. (1) Since, for each normal positive linear form ν, one has $s(\nu) \in M^\nu$, according to Example 8 in the case of normal $\nu, \varrho \in M^*_+$ with equal supports, $s(\nu) = s(\varrho)$, each subalgebra R of M is both $\{\nu, \varrho\}$- and $\{\varrho, \nu\}$-projective.

(2) Obviously, for given $\{\nu, \varrho\}$, the set of all $\{\nu, \varrho\}$-projective subalgebras of M is nonvoid and each subalgebra of a projective subalgebra is projective again. Also, the set of all projective subalgebras of M is closed with respect to intersections.

(3) Suppose $\varrho = \nu^x$, for a pair $\{\nu, \varrho\}$ of normal positive linear forms, with $x \in M_+$ obeying $xs(\varrho) = s(\varrho)x$. Then, according to Example 6 and since (3.6) is obviously fulfilled for $R = R[x]$, the latter subalgebra is an example of a *minimizing Abelian projective subalgebra* of M for $\{\nu, \varrho\}$.

(4) Suppose under the conditions of (3) that a least-minimizing Abelian subalgebra $\mathcal{R}_M(\nu, \varrho)$ exists. According to the previous two items, it follows that $\mathcal{R}_M(\nu, \varrho)$ has to be projective, too.

3.3.2. *Radon–Nikodym Theorem and Minimizing Projective Subalgebras*

For the following recall that in case of $\varrho \ll \nu$ the Radon–Nikodym operator $x = \sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}$ of ϱ relative to ν is understood to be the unique element $x \in M_+$ which obeys both $\varrho = \nu^x$ and $s(x) \leqslant s(\nu)$.

LEMMA 3. *Suppose $\nu, \varrho \in M^*_+$ are normal, with $\varrho \ll \nu$. Let R be any minimizing Abelian projective subalgebra of M for $\{\nu, \varrho\}$. Then the following facts are valid*:

(1) $\forall\, k \in s(\nu)^\perp M_+ s(\nu)^\perp :\; R[\sqrt{d\varrho/d\nu} + k]$ *is minimizing, projective;*

(2) $\exists\, k \in s(\nu)^{\perp} M_{+} s(\nu)^{\perp} :\ R[\sqrt{d\varrho/d\nu} + k] \subset R\,.$

Proof. According to Examples 2 and 6, one knows that the assumptions ensure that minimizing Abelian subalgebras have, in fact, to exist. Since ν is normal, as mentioned above, $I_\nu = Ms(\nu)^{\perp}$ holds. Hence, $I_\nu \cap M_+ = s(\nu)^{\perp} M_+ s(\nu)^{\perp}$ holds and then by Lemma 2 we know that the formula in (1) provides minimizing Abelian subalgebras. Moreover, since $\varrho \ll \nu$ implies $s(\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}) = s(\varrho) \leqslant s(\nu)$, one obviously has that each of $\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + k$, with $k \in s(\nu)^{\perp} M_+ s(\nu)^{\perp}$, commutes with $s(\varrho)$. Hence, by Remark 4(3), all the subalgebras given in accordance with (1) also are projective. Thus, it remains to be shown that each minimizing Abelian projective subalgebra R has a subalgebra as given in line with (1). Note that the assertion holds for $\varrho = 0$ since then $\mathbb{C} \cdot \mathbf{1}$ is minimizing. In line with this, we are going to prove the previous assertion in the nontrivial case with $\nu, \varrho \neq 0$.

Let R be any minimizing Abelian projective subalgebra to the given pair $\{\nu, \varrho\}$. Note that by their very definitions, the conditions of normality for a positive linear form, as well as the relation $\ll$ among normal positive linear forms, are hereditary conditions when considered in restriction to W*-subalgebras of M. Thus, especially we also find $\varrho|_R \ll \nu|_R$ on R. Therefore we have unique Radon–Nikodym operators $x = \sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}$ and $z = \sqrt{\mathrm{d}\varrho|_R/\mathrm{d}\nu|_R}$. As mentioned above, we especially have $s(x) = s(\varrho) \leqslant s(\nu)$ and since $\varrho \neq 0$ is supposed in this case, we also have $z \neq \mathbf{0}$. The assumption that R should be minimizing together with the reasoning of Example 6 when applied for $\{\nu, \varrho\}$ over M, and for $\{\nu|_R, \varrho|_R\}$ over R, respectively, prove that for $x_\varepsilon = x + \varepsilon\,\mathbf{1}$ and $z_\varepsilon = z + \varepsilon\,\mathbf{1}$, with $\varepsilon > 0$, one has

$$\begin{aligned}\lim_{\varepsilon\to 0} \nu(x_\varepsilon) \;&=\; \lim_{\varepsilon\to 0} \varrho(x_\varepsilon^{-1}) = \nu(x) = \sqrt{P_M(\nu,\varrho)} = \sqrt{P_R(\nu|_R, \varrho|_R)}\\ &=\; \nu(z) = \lim_{\varepsilon\to 0} \nu(z_\varepsilon) = \lim_{\varepsilon\to 0} \varrho(z_\varepsilon^{-1}).\end{aligned}$$

Hence, since $\delta = (z_\varepsilon - x_\varepsilon) = (z - x)$ and $\varrho = \nu^x$ hold, upon taking the limit $\varepsilon \to 0$ within the relations which occur if (3.1d) is considered for $z_\varepsilon, x_\varepsilon$ instead of z, x, we will arrive at

$$0 = -\lim_{\varepsilon\to 0} \nu(x x_\varepsilon^{-1}\delta x_\varepsilon^{-1} x) + \lim_{\varepsilon\to 0} \nu(x x_\varepsilon^{-1}\delta z_\varepsilon^{-1}\delta x_\varepsilon^{-1} x)\,, \tag{3.7a}$$

where also the special form of $m(z_\varepsilon, x_\varepsilon)$ arising along with (3.1c) has been taken into account. Also note that by elementary facts on spectral theory, $s_\varepsilon = x x_\varepsilon^{-1} = x_\varepsilon^{-1} x$ is positive for each ε. Also, if positive reals are regarded as a directed set in its descending ordering, then $\{s_\varepsilon\} \subset M_+$ turns into an ascendingly directed net of positive elements of M, with $s_\varepsilon \leqslant s(x)$, and has the support orthoprojection $s(x)$ of x as the least upper bound, that is, l.u.b.$\{s_\varepsilon : \varepsilon > 0\} = s(x)$ is fulfilled. In passing, note that the assertion on monotonicity can be understood as a special consequence of the fact saying that the function $\mathbb{R}_+\backslash\{0\} \ni t \mapsto t^{-1}$ is *operator-(anti)monotoneous* over M_+^{inv} (for generalities on that, see [9, 14]). Since $s(x) = s(\varrho)$ holds, from the previous and with the help of (1.1a) for each $y \in M$, one easily concludes that

$$\begin{aligned}|\nu^{s(\varrho)}(y) - \nu(s_\varepsilon y s_\varepsilon)| &\leqslant |\nu^{s(\varrho)}(y) - \nu(y s_\varepsilon)| + |\nu(y s_\varepsilon) - \nu(s_\varepsilon y s_\varepsilon)| \\ &\leqslant 2\,\|y\|\sqrt{\nu(s(x) - s_\varepsilon)\,\|\nu\|_1}\end{aligned}$$

must be fulfilled. From this, owing to the normality of ν and l.u.b.$\{s_\varepsilon : \varepsilon > 0\} = s(x) = s(\varrho)$,

$$\forall\, y \in M \;:\; \nu^{s(\varrho)}(y) = \lim_{\varepsilon\to 0} \nu(s_\varepsilon y s_\varepsilon) \tag{3.7b}$$

follows. From this and in view of (3.7a) it also follows that both limits within (3.7a) really exist. Now, remember that, by assumption R is both minimizing and projective. Hence, in view of Lemma 1 and Definition 3, both (3.4), with $a = z$ and $p = s(\varrho)$, as well as the particular case of the relation in (3.6) at $y = z$, hold. That is, $\nu(z) = \nu^{s(\varrho)}(z) - \nu^{s(\varrho)^\perp}(z)$ and $\nu(s(\varrho)^\perp z s(\varrho)) = 0$ are fulfilled. From the latter,

$$\nu(z) = \nu^{s(\varrho)}(z) + 2\,\Re\,\nu(s(\varrho)^\perp z s(\varrho)) + \nu^{s(\varrho)^\perp}(z) = \nu^{s(\varrho)}(z) + \nu^{s(\varrho)^\perp}(z)$$

is obtained. This, together with the former, provides the following relation:

$$\nu^{s(\varrho)}(z) = \nu(z)\,. \tag{3.7c}$$

But then, since owing to $s(x) = s(\varrho)$, $\nu^{s(\varrho)}(x) = \nu(x)$ must also be fulfilled and $\nu^{s(\varrho)}(\delta) = \nu(\delta)$ can be followed. Recall that $\nu(\delta) = 0$ holds. In specializing $y = \delta$ within (3.7b), in line with what has been previously stated, (3.7a) can be also read as

$$\lim_{\varepsilon\to 0} \nu(s_\varepsilon \delta z_\varepsilon^{-1} \delta s_\varepsilon) = 0\,. \tag{3.7d}$$

Also note that by the estimate $z_\varepsilon \leqslant (\|z\| + \varepsilon)\,\mathbf{1}$, which is valid by triviality, $(\|z\| + \varepsilon)^{-1}\,\mathbf{1} \leqslant z_\varepsilon^{-1}$ is implied. But then, since the linear map $M \ni y \mapsto s_\varepsilon \delta y \delta s_\varepsilon \in M$ is positive, from the previous and by the positivity of ν, one infers that $\nu(s_\varepsilon \delta z_\varepsilon^{-1} \delta s_\varepsilon) \geqslant (\|z\| + \varepsilon)^{-1} \nu(s_\varepsilon \delta^2 s_\varepsilon) \geqslant 0$. Regarding the limit of the latter as $\varepsilon \to 0$, and respecting that $\|z\| \neq 0$ holds, in view of (3.7d) finally yields $\nu^{s(\varrho)}(\delta^2) = 0$. Owing to $s(\varrho) \leqslant s(\nu)$, from this $\delta s(\varrho) = \mathbf{0}$ follows. Hence, since $s(\varrho) = s(x)$ and $z \in R_+ \subset M_+$ hold, the conclusion is that $z = x + k$ has to be fulfilled, with $k = z s(\varrho)^\perp = s(\varrho)^\perp z \in s(\varrho)^\perp M_+ s(\varrho)^\perp$. But note that, by $\nu(\delta) = 0$, also $\nu(k) = 0$ follows. By the positivity of k and $k s(\varrho)^\perp = k$ from this we conclude that $s(\nu) s(\varrho)^\perp k s(\varrho)^\perp s(\nu) = \mathbf{0}$, which is equivalent to $k s(\varrho)^\perp s(\nu) = \mathbf{0}$, and thus k must obey $k \in s(\nu)^\perp M_+ s(\nu)^\perp$. This, together with the obvious relation $R[x + k] = R[z] \subset R$, is the assertion of (2).

THEOREM 5. *Suppose $\varrho \ll \nu$ is fulfilled, for normal positive linear forms $\nu, \varrho \in M_+^*$, with faithful ν. The following facts hold*:

(1) *provided $\mathcal{R}_M(\nu, \varrho)$ exists it obeys*

$$\mathcal{R}_M(\nu, \varrho) = R\big[\sqrt{d\varrho/d\nu}\big]; \tag{3.8}$$

(2) *if also ϱ is faithful then $\mathcal{R}_M(\nu,\varrho)$ exists.*

Proof. By Lemma 3(1), one knows that $R = R\big[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}\big]$ is minimizing and projective. Hence, if $\mathcal{R}_M(\nu,\varrho)$ is assumed to exist, then by Remark 4(3)–(4), the minimizing subalgebra $\mathcal{R}_M(\nu,\varrho) \subset R\big[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}\big]$ also has to be projective (note that this conclusion does not rely on the premise on faithfulness of ν). Hence, Lemma 3(2) can be applied to $R = \mathcal{R}_M(\nu,\varrho)$. By the faithfulness of ν, one has $s(\nu)^\perp = \mathbf{0}$ and then the mentioned application yields $R \subset R\big[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}\big]$ and, in view of the above, formula (3.8) is seen to hold, that is, (1) is valid. To see (2), note that in this case $\mathbf{1} = s(\nu) = s(\varrho)$ holds, which, via Remark 4(1), implies that Lemma 3(2) can be applied to *each* minimizing R. In line with this, $R\big[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}\big]$ is a minimizing subalgebra of each minimizing R. Thus, it is the least one of this sort.

3.3.3. $\mathcal{R}_M(\nu,\varrho)$ *as a Projective Subalgebra*

Suppose $\varrho \ll \nu$ such that a least-minimizing subalgebra exists. As has been remarked in relation to the previous proof, the algebra $\mathcal{R}_M(\nu,\varrho)$ has to be a minimizing *projective* subalgebra. Application of Lemma 3 then yields that, provided $\mathcal{R}_M(\nu,\varrho)$ exists, the latter has to equal to

$$R_\infty(\nu,\varrho) = \bigcap_{k\in s(\nu)^\perp M_+ s(\nu)^\perp} R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + k]\,. \tag{3.9a}$$

From Lemma 3(2), even $\mathcal{R}_M(\nu,\varrho) = R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + k_\infty]$ can be seen to hold for some $k_\infty \in s(\nu)^\perp M_+ s(\nu)^\perp$. In line with (3.9a), the latter especially means that $R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + k_\infty] \subset R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + \lambda\, s(\nu)^\perp]$ has to be fulfilled for each $\lambda \in \mathbb{R}_+$. Therefore, $k_\infty \in \mathbb{R}_+\, s(\nu)^\perp$ has to hold. In summarizing from the latter and (3.9a), in the general case of $\varrho \ll \nu$ the conclusion of Theorem 5(1) and formula (3.8) generalize to the following implication, which must be fulfilled for some $\gamma \in \mathbb{R}_+$:

$$\begin{aligned}\mathcal{R}_M(\nu,\varrho) \text{ exists} &\Longrightarrow \mathcal{R}_M(\nu,\varrho)\\ &= \bigcap_{\lambda\in\mathbb{R}_+} R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + \lambda\, s(\nu)^\perp]\\ &= R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + \gamma\, s(\nu)^\perp]\\ &= R_\infty(\nu,\varrho)\,.\end{aligned} \tag{3.9b}$$

To summarize from this, for given $\{\nu,\varrho\}$ obeying $\varrho \ll \nu$, the algebra $R_\infty(\nu,\varrho)$ can be regarded to be the only candidate for $\mathcal{R}_M(\nu,\varrho)$. Thereby, the γ within (3.9b) will be made more explicit later.

Note that in the special case of $\varrho \ll \nu$ with $s(\varrho) \in M^\nu$, one can go a step further. Then, since owing to Example 8, the assertion of Lemma 3(2) can be applied to any minimizing subalgebra R, the above can be strengthened to the assertion that, depending whether or not $R_\infty(\nu,\varrho)$ is minimizing, either a least-minimizing Abelian subalgebra will exist which obeys $\mathcal{R}_M(\nu,\varrho) = R_\infty(\nu,\varrho)$, or a least minimizing Abelian subalgebra cannot exist at all.

LEMMA 4. *Suppose $\varrho \ll \nu$, with $s(\varrho) \in M^\nu$. Then $R_\infty(\nu,\varrho)$ is minimizing if and only if a least-minimizing Abelian subalgebra exists.*

Having these facts in mind, and knowing that the special case of faithful ν has been dealt with in Theorem 5, by providing a complete answer for faithful ϱ, we are now going to analyze the family of algebras occurring under the intersection within (3.9b) more thoroughly in the remaining cases (in particular, those with nonfaithful ν) which are not yet covered by the premises of Theorem 5. Some auxiliary technical facts on hereditary subalgebras and elementary spectral theory will be needed for this. Recall some standard facts from W*-theory first.

Remark 5. If $R[y, y^*]$ is the smallest W*-subalgebra of M generated by $y \in M$ and $\mathbf{1}$, then this is the $\sigma(M, M_*)$-closure of all polynomials in $y,\ y^*$ (including the constants as $\mathbb{C}\cdot\mathbf{1}$). Here, M_* is the *predual* of M, which is the Banach (sub)space of M^* (with respect to the functional norm) which is generated by all *normal* positive linear forms (refer also to the elements of M_* as *normal* (*linear*) *forms*). The $\sigma(M, M_*)$-topology is the weakest locally convex topology on M such that all the seminorms $p_f,\ f \in M_*$, with $p_f(x) = |f(x)|$ for $x \in M$, are continuous.

Suppose now $\varrho \ll \nu$, and let an orthoprojection q be defined by $q = s(\varrho) + s(\nu)^\perp$. On the hereditary W*-subalgebra qMq, define another normal positive linear forms ν_q, ϱ_q by $\nu_q = \nu|_{qMq}$ and $\varrho_q = \varrho|_{qMq}$, respectively. Then $\varrho_q \ll \nu_q$ is fulfilled, with supports in qMq obeying $s(\nu_q) = s(\varrho_q) = s(\varrho)$ and $s(\nu_q)^\perp = s(\nu)^\perp$, with '$\perp$' referring to qMq or M, accordingly. Also, if $x = \sqrt{\mathrm{d}\varrho/\mathrm{d}\nu},\ x_q = \sqrt{\mathrm{d}\varrho_q/\mathrm{d}\nu_q}$ are the corresponding Radon–Nikodym operators, one has $x_q = x$ as elements of M. Also, if $\mathrm{spec}_p(x)$ and $\mathrm{spec}_p(x_q)$ are the point-spectra of x and $x_q = x$ with respect to M and qMq, respectively, then the relation

$$\mathrm{spec}_p(x_q) \cup \{0\} = \mathrm{spec}_p(x) \tag{3.10a}$$

can be easily seen to hold. For $y \in (qMq)_\mathrm{h} \subset M_\mathrm{h}$, we let $R_q[y]$ be the W*-subalgebra of qMq generated by y and the unity q of qMq. In view of Remark 5, it is plain to see that $R_q[y] = qR[y]q$ holds. We are going to show that provided $\mathcal{R}_M(\nu,\varrho)$ exists, then $\mathcal{R}_{qMq}(\nu_q,\varrho_q)$ exists, and obeys

$$\mathcal{R}_{qMq}(\nu_q,\varrho_q) = q\mathcal{R}_M(\nu,\varrho)q\,. \tag{3.10b}$$

In fact, since owing to $s(x) = s(\varrho)$ for each $k \in s(\nu)^\perp M s(\nu)^\perp$ also $x + k \in qMq$ holds, one has $R_q[x_q + k] = qR[x+k]q$. Hence, in accordance with (3.9a) and (3.9b), one has

$$q\mathcal{R}_M(\nu,\varrho)q = \bigcap_{\lambda\geqslant 0} R_q[x_q + \lambda\, s(\nu)^\perp] = R_q[x_q + \gamma\, s(\nu)^\perp] = \bigcap_k R_q[x_q + k],$$

for some real $\gamma \geqslant 0$. We may apply formula (3.9a) with respect to the hereditary algebra qMq and normal positive linear forms ν_q, ϱ_q. The result is $R_\infty(\nu_q, \varrho_q) = \bigcap_k R_q[x_q + k]$, with k running through $s(\nu_q)^\perp M_+ s(\nu_q)^\perp = s(\nu)^\perp M_+ s(\nu)^\perp$ (see above). Hence, in view of the previous, one has

$$q\mathcal{R}_M(\nu, \varrho)q = R_q[x_q + \gamma\, s(\nu)^\perp] = R_\infty(\nu_q, \varrho_q).$$

Especially, application of Lemma 3(1) for ν_q, ϱ_q on qMq shows that $R_\infty(\nu_q, \varrho_q)$ is minimizing. But, since $s(\nu_q) = s(\varrho_q)$ and $\varrho_q \ll \nu_q$ hold, when considering Lemma 4, Remark 4(1) and (3.9b) for ν_q, ϱ_q on qMq, one gets $R_\infty(\nu_q, \varrho_q) = \mathcal{R}_{qMq}(\nu_q, \varrho_q)$. From this, in view of the above, (3.10b) follows.

We close our preliminaries with the following auxiliary result which matters in some elementary spectral theory.

LEMMA 5. *Suppose* $x \in M_+$, $s(x) < \mathbf{1}$, *with point spectrum* $\operatorname{spec}_p(x)$. *Depending on the latter, the following cases may occur for the commutative* W*-*subalgebra* $R_0(x) = \bigcap_{\lambda \in \mathbb{R}_+} R[x + \lambda\, s(x)^\perp]$, *where* γ *can stand for any nonnegative real*:

$$R_0(x) \begin{cases} = R[x] & \textit{if } \operatorname{spec}_p(x)\backslash\{0\} = \emptyset, \\ = R[x + \lambda_0\, s(x)^\perp] & \textit{if } \operatorname{spec}_p(x)\backslash\{0\} = \{\lambda_0\}, \\ \neq R[x + \gamma\, s(x)^\perp] & \textit{if } \#\operatorname{spec}_p(x)\backslash\{0\} \geqslant 2. \end{cases}$$

Especially, $R_0(x) = R[x]$ *holds if and only if* $\operatorname{spec}_p(x)\backslash\{0\} = \emptyset$ *is fulfilled.*

Proof. Some preliminary results will be derived first. Let $\{E_x(t) : t \in \mathbb{R}\}$ be the spectral resolution of x within the projection lattice of M. Then the eigenprojection of the positive element $x + \lambda\, s(x)^\perp$ to the spectral value $\lambda \in \mathbb{R}_+$ is given by

$$E_{x+\lambda\, s(x)^\perp}(\{\lambda\}) = \begin{cases} s(x)^\perp + E_x(\{\lambda\}) & \text{for } \lambda \in \mathbb{R}_+\backslash\{0\}\,, \\ s(x)^\perp & \text{for } \lambda = 0\,. \end{cases} \tag{$\star$}$$

In fact, by assumption, $E_x(\{0\}) = s(x)^\perp$ holds and thus the part of ($\star$) relating to $\lambda = 0$ is valid. Also, for $\lambda \in \mathbb{R}_+\backslash\{0\}$ it is clear from $E_x(\{0\})E_x(\{\lambda\}) = \mathbf{0}$ and the above that $p = s(x)^\perp + E_x(\{\lambda\})$ is an orthoprojection in M which obeys $(x + \lambda\, s(x)^\perp)p = \lambda\, p$. Note, within this context, that $E_x(\{\lambda\})$ is nonvanishing iff $\lambda \in \operatorname{spec}_p(x)$. Also, for an orthoprojection $q \geqslant p$, one has $(q - p)\, s(x)^\perp = \mathbf{0}$ and $(q - p)\, E_x(\{\lambda\}) = \mathbf{0}$. Hence, assuming $(x + \lambda\, s(x)^\perp)\, q = \lambda\, q$ yields $x\,(q - p) = \lambda\,(q - p)$, which according to spectral theory necessarily implies $(q - p) \leqslant E_x(\{\lambda\})$. In view of the above, $(q - p) = \mathbf{0}$. Thus, there is no larger than p orthoprojection q in M with $(x + \lambda\, s(x)^\perp)\, q = \lambda\, q$, which means $p = E_{x+\lambda\, s(x)^\perp}(\{\lambda\})$. This is ($\star$).

Next, it is useful to note that the following alternatives exist:

$$R[x + \lambda\, s(x)^\perp] \begin{cases} = R[x] & \text{if } \lambda \notin \operatorname{spec}_p(x)\backslash\{0\} \text{ or } \lambda = 0, \\ \subsetneqq R[x] & \text{else.} \end{cases} \tag{$\star\star$}$$

To see ($\star\star$), first note that obviously $R[x+\lambda\, s(x)^{\perp}]\subset R[x]$. Since, for $\lambda\notin \mathrm{spec}_p(x)\backslash\{0\}$, one has $E_x(\{\lambda\})=\mathbf{0}$, from ($\star$), then $E_{x+\lambda\, s(x)^{\perp}}(\{\lambda\})=s(x)^{\perp}$ is seen and thus both $x+\lambda\, s(x)^{\perp}$ and $s(x)^{\perp}$ have to belong to $R[x+\lambda\, s(x)^{\perp}]$, and so does x. In view of the above, $R[x+\lambda\, s(x)^{\perp}]=R[x]$, which is trivially valid for $\lambda=0$, is seen to hold for $\lambda\notin\mathrm{spec}_p(x)\backslash\{0\}$. In the case of $\lambda\in \mathrm{spec}_p(x)\backslash\{0\}$, the element $x+\lambda\, s(x)^{\perp}$ has full support and, according to ($\star$), $s(x)^{\perp}$ is a *proper* subprojection of the eigenorthoprojection $E_{x+\lambda\, s(x)^{\perp}}(\{\lambda\})$ to the spectral value $\lambda\in\mathrm{spec}_p(x+\lambda\, s(x)^{\perp})$. Since each spectral eigenprojection has to be a minimal orthoprojection of the generated commutative W*-algebra $R[x+\lambda\, s(x)^{\perp}]$, from the previous $s(x)^{\perp}\notin R[x+\lambda\, s(x)^{\perp}]$ has to be followed. Hence, in this case, $R[x+\lambda\, s(x)^{\perp}]\subsetneqq R[x]$, which completes the proof of ($\star\star$).

After these preparations, we are going to prove the assertions of our results on $R_0(x)$. Note that the validity in the case of $\mathrm{spec}_p(x)\backslash\{0\}=\emptyset$ or $\mathrm{spec}_p(x)\backslash\{0\}=\{\lambda_0\}$ is straightforward from ($\star\star$). Thus, we have to explicitly consider only the case with $\#\mathrm{spec}_p(x)\backslash\{0\}\geqslant 2$. From ($\star\star$), $R_0(x)\subsetneqq R[x]$ obviously follows. Especially this also means that the assertion is valid for $\gamma=0$. Now, in line with this, but in contrast with the assertion, we assume $R_0(x)=R[x+\gamma\, s(x)^{\perp}]$, with $\gamma>0$. Then, since $\#\mathrm{spec}_p(x)\backslash\{0\}\geqslant 2$ is fulfilled, there has to exist $\lambda\in\mathrm{spec}_p(x)\backslash\{0\}$ with $\lambda\neq\gamma$. Thus,

$$E_{x+\lambda\, s(x)^{\perp}}(\{\lambda\})\in R[x+\lambda\, s(x)^{\perp}],\quad\text{and}\quad E_{x+\gamma\, s(x)^{\perp}}(\{\gamma\})\in R_0(x)$$

by assumption. Since by definition of $R_0(x)$ one has $R_0(x)\subset R[x+\lambda\, s(x)^{\perp}]$, both $E_{x+\lambda\, s(x)^{\perp}}(\{\lambda\})$ and $E_{x+\gamma\, s(x)^{\perp}}(\{\gamma\})$ have to be in $R[x+\lambda\, s(x)^{\perp}]$. From ($\star$) and since $\gamma\neq\lambda$ is fulfilled, we see that

$$s(x)^{\perp}=E_{x+\gamma\, s(x)^{\perp}}(\{\gamma\})\, E_{x+\lambda\, s(x)^{\perp}}(\{\lambda\})\in R[x+\lambda\, s(x)^{\perp}]$$

and therefore also $x\in R[x+\lambda\, s(x)^{\perp}]$ holds. From this and $R[x+\lambda\, s(x)^{\perp}]\subset R[x]$, $R[x+\lambda\, s(x)^{\perp}]=R[x]$ had to be followed. Owing to the choice of λ in accordance with $\lambda\in\mathrm{spec}_p(x)\backslash\{0\}$, this is in contradiction with ($\star\star$). Thus, also in the case of $\gamma>0$, a relation $R_0(x)=R[x+\gamma\, s(x)^{\perp}]$ cannot happen. Finally, note that by the just proven, allowance is made for any situations with $R_0(x)$ that might occur. Particularly, from this and ($\star\star$), one also infers that $R_0(x)=R[x]$ cannot happen unless $\mathrm{spec}_p(x)\backslash\{0\}=\emptyset$, whereas in the latter case this then, in fact, occurs. Thus, also the final assertion is seen to be true.

3.3.4. *The Main Result for $\varrho\ll\nu$ and with $s(\varrho)\in M^{\nu}$*

Suppose $\varrho\ll\nu$ such that $\mathcal{R}_M(\nu,\varrho)$ exists. Then we derive a formula of $\mathcal{R}_M(\nu,\varrho)$ which generalizes (3.8) to this context. In addition, partial answers on the existence problem for $\mathcal{R}_M(\nu,\varrho)$ will be also given.

THEOREM 6. *Let M be a* W*-*algebra, and let two normal positive linear forms ν,ϱ be given on M and obeying $\varrho\ll\nu$. Let a nonnegative real λ_0 be defined by*

$$\lambda_0=\sup\left\{\lambda:\ \lambda\in\mathrm{spec}_p(\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu})\cup\{0\}\right\}. \tag{3.11a}$$

The following facts hold true.

(1) *Provided* $\mathcal{R}_M(\nu,\varrho)$ *exists, it obeys*

$$\mathcal{R}_M(\nu,\varrho) = R\big[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} + \lambda_0\, s(\nu)^\perp\big], \tag{3.11b}$$

with the additional condition

$$\#\mathrm{spec}_p(\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}) \begin{cases} \leqslant 2 & \textit{if } s(\varrho) = s(\nu), \\ = 1 & \textit{else} \end{cases} \tag{3.11c}$$

fulfilled in the case of nonfaithful ν.

(2) *Assume* $\{\nu,\varrho\}$ *with* $s(\varrho) \in M^\nu$. *Then, if* ν *is faithful, or in all cases with nonfaithful* ν *obeying* $\dim s(\nu)^\perp M s(\nu)^\perp < \infty$ *and* ϱ *respecting* (3.11c), *a least minimizing Abelian subalgebra exists.*

Proof. Let $x = \sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}$ and assume that $\mathcal{R}_M(\nu,\varrho)$ exists. Then (3.9b) yields that $\mathcal{R}_M(\nu,\varrho) = R[x+\gamma\, s(\nu)^\perp]$ has to be fulfilled for some $\gamma \in \mathbb{R}_+$. We are going to determine the real γ in terms of x. Let $q = s(\varrho) + s(\nu)^\perp$. According to (3.10b) and by using the notations introduced within the context of Equations (3.10), with respect to the hereditary W*-subalgebra qMq and normal positive linear forms ν_q, ϱ_q, $\mathcal{R}_{qMq}(\nu_q,\varrho_q)$ also exists and obeys $\mathcal{R}_{qMq}(\nu_q,\varrho_q) = R_q[x_q + \gamma\, s(\nu)^\perp]$. On the other hand, an application of (3.9b) on qMq with ν_q, ϱ_q yields $\mathcal{R}_{qMq}(\nu_q,\varrho_q) = R_0(x_q)$, with the algebra $R_0(x_q)$ constructed as in Lemma 5 in terms of $x_q = \sqrt{\mathrm{d}\varrho_q/\mathrm{d}\nu_q}$ and with respect to qMq. Since both $s(x_q) = s(\varrho_q) = s(\nu_q) = s(\varrho)$ and $s(\nu_q)^\perp = s(\nu)^\perp$ hold on qMq, in view of the above, we therefore conclude that, provided $\mathcal{R}_M(\nu,\varrho)$ has been assumed to exist, then $R_0(x_q) = R_q[x_q + \gamma\, s(x_q)^\perp]$ has to be fulfilled for some $\gamma \in \mathbb{R}_+$. But then, in the case of $s(\varrho) = s(x_q) < q$, Lemma 5 can be applied on qMq and gives that $\#\mathrm{spec}_p(x_q)\backslash\{0\} < 2$ has to be fulfilled, with $\gamma = \sup\{\lambda : \lambda \in \mathrm{spec}_p(x_q) \cup \{0\}\}$. Note that the condition $s(\varrho) = s(x_q) < q$ is equivalent to $s(\nu) < \mathbf{1}$, and that in this case, $0 \in \mathrm{spec}_p(x)$ holds. Hence, by (3.10a) in this case $\#\mathrm{spec}_p(x_q)\backslash\{0\} = \#\mathrm{spec}_p(x)\backslash\{0\}$. Especially, the previously given γ then obeys $\gamma = \lambda_0$, with λ_0 as given in accordance with (3.11a). Thus, in summarizing from this and the previous, and assuming that $\mathcal{R}_M(\nu,\varrho)$ exists for nonfaithful ν, implies that (3.11b) and $\#\mathrm{spec}_p(x) \leqslant 2$ hold. Now, suppose $s(\varrho) < s(\nu) < \mathbf{1}$. Then, assuming $\lambda_0 > 0$ would imply $q^\perp \in R[x + \lambda_0\, s(\nu)^\perp]$, for $q^\perp$ is the eigenprojection of $x + \lambda_0\, s(\nu)^\perp$ to eigenvalue 0. But at the same time, certainly $q^\perp \notin R[x]$ since by supposition of this case, $q^\perp < s(\varrho)^\perp$ has to hold and $s(\varrho)^\perp$ has to be a minimal orthoprojection of $R[x]$. Thus, $R[x + \lambda_0\, s(\nu)^\perp]$ cannot be a subalgebra of $R[x]$ in this case. In view of the meaning of $\mathcal{R}_M(\nu,\varrho)$ and since $R[x]$ is minimizing, the latter contradicts the just-derived formula (3.11b) in the case of nonfaithful ν. Hence, for $s(\varrho) < s(\nu) < \mathbf{1}$, one must have $\lambda_0 = 0$. In view of (3.11a) and since for nonfaithful ν, one has $0 \in \mathrm{spec}_p(x)$ it is then inferred that $\mathrm{spec}_p(x) = \{0\}$ holds. This completes the proof of (3.11c). That (3.11b) remains true also for faithful ν follows since, owing to $s(\nu)^\perp = \mathbf{0}$, formula (3.11b) simply reduces to formula (3.8), which according to Theorem 5(1) is true, however, and completes the proof of (1).

To see (2), note that for faithful ν, formula (3.9a) yields $R_\infty(\nu,\varrho) = R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}]$. Hence, according to Lemma 3, the algebra $R_\infty(\nu,\varrho)$ is minimizing. But then, since ϱ obeys $s(\varrho) \in M^\nu$, from Lemma 4 we may also conclude that $\mathcal{R}_M(\nu,\varrho)$ exists. This proves the part of (2) relating to a faithful ν.

Suppose now that ν is nonfaithful, but with $\dim s(\nu)^\perp M s(\nu)^\perp < \infty$ fulfilled, and ϱ such that $s(\varrho) \in M^\nu$ holds and condition (3.11c) is respected. Note that $0 \in \mathrm{spec}_p(x)$ holds in this case. Also, by the assumption of finite-dimensionality, $\mathrm{spec}(k) = \mathrm{spec}_p(k)$ holds for each $k \in s(\nu)^\perp M_+ s(\nu)^\perp$, and if p_λ is the eigenprojection of k to $\lambda \in \mathrm{spec}_p(k)$, we have $\sum_{\lambda\in\mathrm{spec}_p(k)} p_\lambda = s(\nu)^\perp$. By the same kind of auxiliary arguments as for elementary spectral theory, which have been used in the proof of Lemma 5 in some special case, in literally the same way (the details of which therefore will not be mentioned) can also be applied in order to compare the spectral structures of $x+k$ and x (these facts will be tacitly made use of below). Suppose $\lambda_0 = 0$ first. Then, zero is the only eigenvalue of x, and therefore one infers that $\mathrm{spec}_p(x+k) = \mathrm{spec}_p(k) \cup \{0\}$ for $s(\varrho) < s(\nu)$, and $\mathrm{spec}_p(x+k) = \mathrm{spec}_p(k)$ for $s(\varrho) = s(\nu)$. Owing to this and to $s(x) \leqslant s(\nu)$, whereas each of the above p_λ for $\lambda \in \mathrm{spec}_p(k)\backslash\{0\}$ will be also the corresponding eigenprojection to the same $\lambda \in \mathrm{spec}_p(x+k)$ with respect to $x+k$, the projection $p_0 + \{s(\nu) - s(\varrho)\}$, or $\{s(\nu) - s(\varrho)\}$ respectively, will be the eigenprojection of $x+k$ to the eigenvalue zero in the case of $0 \in \mathrm{spec}_p(x+k) \cap \mathrm{spec}_p(k)$, and in the case of $0 \in \mathrm{spec}_p(x+k)$ but with $0 \notin \mathrm{spec}_p(k)$, respectively. Therefore, $p_\lambda \in R[x+k]$ for each $\lambda \in \mathrm{spec}_p(k)\backslash\{0\}$, and $p_0 + \{s(\nu) - s(\varrho)\} \in R[x+k]$ in the case of $0 \in \mathrm{spec}_p(x+k) \cap \mathrm{spec}_p(k)$ or $\{s(\nu) - s(\varrho)\} \in R[x+k]$ in the case of $0 \in \mathrm{spec}_p(x+k)$ but with $0 \notin \mathrm{spec}_p(k)$. But then in view of the above, in each case, also their sum $s(\nu)^\perp + \{s(\nu) - s(\varrho)\}$ has to be in $R[x+k]$, that is, $s(\varrho)^\perp \in R[x+k]$ has to hold. From this and owing to $s(x) = s(\varrho) \leqslant s(\nu)$, $x = s(\varrho)\{x+k\} \in R[x+k]$ is seen. Hence, $R[x+k] \supset R[x]$ follows for each $k \in s(\nu)^\perp M_+ s(\nu)^\perp$ and, therefore, one has $R_\infty(\nu,\varrho) = R[x]$. From Lemma 3 it follows that $R_\infty(\nu,\varrho)$ is minimizing. Thus, since ϱ obeys $s(\varrho) \in M^\nu$, we may conclude from Lemma 4 that $\mathcal{R}_M(\nu,\varrho)$ exists. Hence, for nonfaithful ν and $\#\mathrm{spec}_p(x) = 1$, the assertion of (2) is true.

Suppose $s(\varrho) = s(\nu)$ and $\#\mathrm{spec}_p(x) = 2$, with nonfaithful ν. Then, $\lambda_0 > 0$, and for each $k \in s(\nu)^\perp M_+ s(\nu)^\perp$, one has $p_\lambda \in R[x+k]$ for $\lambda \in \mathrm{spec}_p(k)\backslash\{\lambda_0\}$. If $\lambda_0 \notin \mathrm{spec}_p(k)$, $s(\nu)^\perp \in R[x+k]$ follows from this, and so $R[x] \subset R[x+k]$ is seen. For $\lambda_0 \in \mathrm{spec}_p(k)$, however, $p_{\lambda_0} + E_x(\{\lambda_0\})$ is the λ_0 corresponding eigenprojection of $x+k$, and therefore instead of $p_{\lambda_0} \in R[x+k]$, one finds $p_{\lambda_0} + E_x(\{\lambda_0\}) \in R[x+k]$. Summing up yields $s(\nu)^\perp + E_x(\{\lambda_0\}) \in R[x+k]$ instead. But then also

$$k + \lambda_0\, E_x(\{\lambda_0\}) = (s(\nu)^\perp + E_x(\{\lambda_0\}))(x+k) \in R[x+k].$$

Hence, since $x + \lambda_0\, s(\nu)^\perp$ can be combined together from the mentioned elements as

$$x + \lambda_0\, s(\nu)^\perp = (x+k) - (k + \lambda_0\, E_x(\{\lambda_0\})) + \lambda_0(s(\nu)^\perp + E_x(\{\lambda_0\})),$$

$$x + \lambda_0\, s(\nu)^{\perp} \in R[x + k]$$

is seen. Note that owing to $s(\nu)^{\perp} \in R[x]$, in any case one has $x + \lambda_0\, s(\nu)^{\perp} \in R[x]$. We may summarize these facts and conclude that, for nonfaithful ν with $s(\varrho) = s(\nu)$ and $\#\mathrm{spec}_p(x) = 2$, $R[x + \lambda_0\, s(\nu)^{\perp}] \subset R[x + k]$ holds for each $k \in s(\nu)^{\perp} M_+ s(\nu)^{\perp}$. Hence, $R_\infty(\nu, \varrho) = R[x + \lambda_0\, s(\nu)^{\perp}]$, and thus according to Lemma 3, in this case the algebra $R_\infty(\nu, \varrho)$ is also minimizing. Since $s(\varrho) = s(\nu) \in M^\nu$ holds, Lemma 4 can be applied once more and yields that $\mathcal{R}_M(\nu, \varrho)$ exists. This closes the proof of (2) and at the same time also completes the proof of the theorem.

3.3.5. *Examples and Consequences*

We start by discussing Theorem 6 in the finite-dimensional case.

EXAMPLE 9. Suppose that $2 \leqslant \dim M < \infty$, and ν, ϱ are two nonzero positive linear forms obeying $\varrho \ll \nu$, but which are not mutually proportional. Then, the corresponding Radon–Nikodym operator cannot be proportional to the support of ν, $\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} \notin \mathbb{R}_+\, s(\nu)$. Since $s(\varrho) \leqslant s(\nu)$ is the support of $\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}$, from these facts $\#\,\mathrm{spec}(\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}) \geqslant 2$ follows. Hence, since by finite-dimensionality, one has $\mathrm{spec}_p(\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}) = \mathrm{spec}(\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu})$, the condition (3.11c) in the case of nonfaithful ν could be satisfied only if $\#\,\mathrm{spec}(\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}) = 2$ and $s(\varrho) = s(\nu) < \mathbf{1}$ were fulfilled. But then $\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}$, as a Radon–Nikodym operator had to be proportional with $s(\varrho) = s(\nu)$, which contradicts the above-mentioned fact. Thus, in view of Theorem 6(1) for nonfaithful ν and under the above premises, a least-minimizing algebra cannot exist in the finite-dimensional case. Especially, from the latter and by formula (3.11b), one also infers that, provided a least-minimizing algebra exists, $\mathcal{R}_M(\nu, \varrho) = R[\sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}]$ will occur, in any case. From Theorem 6(2), one infers that the latter case really can happen, e.g. in the case of faithful ν and ϱ obeying $\varrho \ll \nu$ and $s(\varrho) \in M^\nu$.

As the previous example shows, the deviation from the law (3.8) as indicated by (3.11b) could be observed only for $\dim M = \infty$. That this deviation really can occur is seen from the following example.

EXAMPLE 10. Let $M = \mathsf{L}^\infty(I, m')$, where $\{I, m'\}$ is the unit interval $I = [0, 1]$ with a measure $m' = (m + \delta_0)/2$, where m is the Lebesgue measure and δ_0 is concentrated on $\{0\}$, with $\delta_0(\{0\}) = 1$. Let ν correspond to the class of the characteristic function $\chi_{(0,1]}$ of $(0, 1]$ via $\nu(\cdot) = \int_{(0,1]} (\cdot)\, \mathrm{d}m'$, and be f a strictly increasing function, which is continuous on $[0, 1]$, except for one point $t_0 > 0$ where it is only left-continuous with $f(t_0) = \lambda_0 > 0$, and which obeys $0 < f(t) \leqslant 1$ for $t > 0$, and $f(0) = 0$. Define $\varrho(\cdot) = \int_I (\cdot) f\, \mathrm{d}m'$. Then, $\varrho \ll \nu$ (even $\varrho \leqslant \nu$ holds) and $s(\nu) = s(\varrho) = \chi_{(0,1]} < \chi_{[0,1]} = \mathbf{1}$, with Radon–Nikodym operator $x = f$ obeying

$\{0, \lambda_0\} = \mathrm{spec}_p(x)$. Hence, condition (3.11c) is fulfilled in this case. Since, owing to $M^\nu = M$, one has $s(\varrho) \in M^\nu$ to be fulfilled by triviality, Theorem 6(2) can be applied and formula (3.11b) then yields $\mathcal{R}_M(\nu, \varrho) = R[f + \lambda_0\, \chi_{\{0\}}]$.

Along with Theorem 6(1) comes another necessary condition for $\mathcal{R}_M(\nu, \varrho)$ to exist which will often be useful. To explain this, in the following let $\mathsf{Aut}(M)$ denote the group of all $*$-automorphisms of M, and for $y \in M$ we let $\mathsf{Aut}_y(M)$ be those $*$-automorphisms which leave the element y fixed. Clearly, since we have to do with $*$-automorphisms, one has $\mathsf{Aut}_y(M) = \mathsf{Aut}_{y^*}(M)$, for each $y \in M$.

Remark 6. Recall that a $*$-isomorphism Φ from one W^*-algebra M onto another W^*-algebra N is automatically $\sigma(M, M_*)$-$\sigma(N, N_*)$ continuous. From this and Remark 5 follows that $\Phi \in \mathsf{Aut}_y(M) \iff \Phi \in \mathsf{Aut}_x(M),\ \forall\, x \in R[y, y^*]$, is valid for each $y \in M$.

COROLLARY 5. *For the pair $\{\nu, \varrho\}$ of normal positive linear forms suppose $\varrho \ll \nu$, with Radon–Nikodym operator $x = \sqrt{\mathrm{d}\varrho/\mathrm{d}\nu}$, and let λ_0 be defined in accordance with formula* (3.11a). *Then the existence of $\mathcal{R}_M(\nu, \varrho)$ implies that the following holds*

$$\forall\, k \in s(\nu)^\perp M_+ s(\nu)^\perp :\ \mathsf{Aut}_{x+k}(M) \subset \mathsf{Aut}_{x+\lambda_0\, s(\nu)^\perp}(M)\,. \tag{3.12}$$

Proof. In view of (3.9a) and Theorem 6(1), the premises imply $R[x+\lambda_0\, s(\nu)^\perp] \subset R[x + k]$ to be fulfilled for each $k \in s(\nu)^\perp M_+ s(\nu)^\perp$. From this, it is evident that by each $*$-automorphisms Φ leaving pointwise invariant all elements of $R[x + k]$, in particular also each element of $R[x + \lambda_0\, s(\nu)^\perp]$ is left invariant. This is (3.12).

We will show that among the assumptions in Theorem 6(2), also the condition $\dim s(\nu)^\perp M s(\nu)^\perp < \infty$ is a sensitive one. For simplicity, this will be demonstrated by such an example which, by its construction and owing to the procedure applied, can stand for a whole class of analogous (even noncommutative) situations where (3.12) fails and thus a least-minimizing subalgebra cannot exist then.

EXAMPLE 11. Let $M = \mathsf{L}^\infty(I, m)$, where $\{I, m\}$ is the unit interval $I = [0, 1]$ with Lebesgue measure m. Let $\tau \in M_+^*$ be the standard tracial state given on M by $\tau(x) = \int_I \mathrm{d}m\, x$, for $x \in M$. Suppose $\nu = \tau(\chi_0(\cdot))$, where χ_0 corresponds to the class of the characteristic function of the interval $[0, 1/2]$. Assume $\varrho = \tau(f(\cdot))$, where we let f correspond to the class of some continuous, monotoneous function f on $[0, 1]$, with $1 \geqslant f(t) > 0$ for $t < 1/2$ and $f(t) = 0$ else. We then have $\varrho \ll \nu$, $s(\nu) = \chi_0 < \mathbf{1}$ and $x = \sqrt{\mathrm{d}\varrho/\mathrm{d}\nu} = f$. Let us consider the *-automorphism Φ_g which is induced on M by the measure-preserving point-transformation $g : I \ni t \mapsto (1 - t) \in I$ of the unit interval, that is, in the sense of the equivalence of functions, $\Phi_g(x) = x \circ g$ is fulfilled. Obviously, Φ_g is idempotent, that is, a symmetry. Note that $\Phi_g(\chi_0) = \chi_1$ holds, where χ_1 stands for the class of the characteristic function of the interval $[1/2, 1]$ within M, that is, $\Phi_g(\chi_0) = \chi_0^\perp$ is fulfilled. From $0 \leqslant f \leqslant \chi_0$, $\Phi_g(f) \in \chi_0^\perp M_+ \chi_0^\perp$ follows. Let us define $k = \Phi_g(f)$. Owing to idempotency of Φ_g, $\Phi_g \in \mathsf{Aut}_{x+k}(M)$ follows. On the other hand, according to the above and since $\chi_0 \in R[x]$ holds, we certainly must have $\Phi_g \notin \mathsf{Aut}_x(M)$. In fact, otherwise according to the equivalence mentioned in Remark 6, in contrast to the above we also had χ_0 to be a fixed point of Φ_g, a contradiction. Now, the Radon-Nikodym operator $x = f$ by choice of f obeys $\mathrm{spec}_p(x) = \{0\}$. Hence, $\lambda_0 = 0$. But then the existence of the above constructed Φ_g proves that condition (3.12) is violated, and thus in view of Corollary 5, this means that $\mathcal{R}_M(\nu, \varrho)$ cannot exist in the case to hand.

3.3.6. *Does Each Minimizing Subalgebra Dominate a Minimizing Projective Subalgebra?*

Note that, according to Theorem 6(1) and Lemma 3(1), the existence of the least-minimizing subalgebra also means that each minimizing subalgebra R possesses a minimizing projective subalgebra. One finds the following useful auxiliary characterization of this fact:

COROLLARY 6. *Let ν, ϱ be normal positive linear forms with $\varrho \ll \nu$ and Radon–Nikodym operator x. Let R be a minimizing Abelian* W**-subalgebra, and let $z \in R_+$ be the R-relative Radon–Nikodym operator achieving $\varrho|_R = \nu|_R^z$. The following items are mutually equivalent*:

(1) *$R_1 \subset R$, for some minimizing projective subalgebra R_1;*
(2) $\nu^{s(\varrho)}(z) = \nu(z)$.

In the latter case, $R_1 = R[x+k]$ can be chosen in (1) *for some $k \in s(\nu)^\perp M_+ s(\nu)^\perp$.*

Proof. For a minimizing R, the condition $\nu^{s(\varrho)}(z) = \nu(z)$ implies, the existence of $k \in s(\nu)^\perp M_+ s(\nu)^\perp$ with $R[x + k] \subset R$. This can be seen exactly in the same way as demonstrated in the course of the proof of Lemma 3(2) (see from (3.7c) onward). In view of Lemma 3(1), $R_1 = R[x+k]$ can be chosen from (1). To see the other direction, assume $R_1 \subset R$ with some minimizing projective subalgebra R_1. From Lemma 3(2), one knows that $k \in s(\nu)^\perp M_+ s(\nu)^\perp$ exists with $R[x + k] \subset R_1$. Then $R[x + k] \subset R$ also holds, and thus $x + k \in R$. Owing to $s(\varrho|_R) \in R$ and since R is commutative, one has $y = s(\varrho|_R)(x + k) = (x + k)s(\varrho|_R) \in R_+$. From this and $\varrho = \nu^x = \nu^{(x+k)}$, then $\varrho|_R = \nu^{(x+k)}|_R = \nu^{(x+k)s(\varrho|_R)}|_R = \nu^y|_R = \nu|_R^y$ is seen. In view of $s(y) \leqslant s(\varrho|_R)$ and by the uniqueness of the Radon–Nikodym operator z in R, $z = y$ follows. Now, $s(\varrho|_R) \geqslant s(\varrho)$ and $s(x) = s(\varrho) \leqslant s(\nu)$ hold. Hence, $s(\varrho)z = s(\varrho)y = s(\varrho)s(\varrho|_R)(x + k) = s(\varrho)(x + k) = x$ must be fulfilled, and therefore also $s(\varrho)z = zs(\varrho) = s(\varrho)zs(\varrho)$. Since R is minimizing, from the previous, together with Lemma 1 (put $p = s(\varrho)$ and $a = z$ in (3.4)) by literally the same arguments which led us to see (3.7c) within the proof of Lemma 3(2) the desired relation $\nu^{s(\varrho)}(z) = \nu(z)$ is seen to also hold in the situation to hand.

Remark 7. (1) The condition $s(\varrho) \in M^{\nu}$ within Theorem 6(2) makes that Corollary 6(2) is trivially satisfied and then, in line with Remark 4(1), each minimizing subalgebra is projective.

(2) Suppose $\varrho \ll \nu$ but with $s(\varrho) \notin M^{\nu}$ (thus M cannot be commutative). It is an open question whether other minimizing subalgebras than those respecting Corollary 6(2) could exist at all.

(3) Note that $M = \mathsf{M}_2(\mathbb{C})$ is the least case where the previous question might be nontrivial (cf. Example 9). But in this case, the characteristic configuration of a pair $\{\nu, \varrho\}$ to be dealt with for a decision in the usual canonical manner, may be reduced to pairs $\{a, p\}$ of 2×2-matrices, with positive definite a and one-dimensional orthoprojection p obeying $pa \neq ap$. Thus, calculations can be carried out explicitly (we omit the details) and, in fact, show that $R = R[x] = R[p]$ is the only minimizing subalgebra. This also completes the analysis of Example 9 in the 2×2 case: for ν, ϱ which are not mutually proportional and which obey $\varrho \ll \nu$ the least minimizing subalgebra exists iff ν is faithful. In view of Example 7, it follows that, for a general pair of mutually nonproportional positive linear forms on $M = \mathsf{M}_2(\mathbb{C})$, $\mathcal{R}_M(\nu, \varrho)$ exists if and only if at least one of the two forms is faithful. Thus, in this case we have a complete solution of the problem for a noncommutative M, even without imposing the condition $\varrho \ll \nu$.

(4) Suppose $\{\nu, \varrho\}$ such that Corollary 6(2) is fulfilled in each case of a minimizing subalgebra. Then the problem of the existence of a least-minimizing subalgebra will be reduced to the question of whether or not $R_{\infty}(\nu, \varrho)$ were equal to $R[x + \lambda_0\, s(\nu)^{\perp}]$ (see Lemma 4 for a special case). As Example 11 shows, for the latter to happen both (3.11c) and (3.12) are necessary conditions and are rather independent from each other.

(5) The method by means of which the assertion on equality of the intersection algebra $R_{\infty}(\nu, \varrho)$ of (3.9a) to one of the intersecting minimizing subalgebras $R[x+\lambda_0\, s(\nu)^{\perp}]$ has been disproved, and which is based on considering symmetries, seems to be very effective and in a modified form is a common method to disprove the uniqueness of optimizing elements (algebras, decompositions, etc.) in similar $*$-algebraic optimization problems, see, e.g., [28].

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