Statistical Physics II
Problem Set 9

Due: Tuesday, June 10, **before** the lecture

13. **1-dimensional Ising model** **(4 points)**

The Hamiltonian of the one-dimensional Ising model for N spins ($s_i = \pm 1$, $i = 1, \ldots, N$) with open boundary conditions is given by

$$H = -\sum_{i=1}^{N-1} J_i s_i s_{i+1} \,.$$

J_i is the interaction energy of the neighboring spins at i and $i+1$.

a) Calculate the partition sum

$$Z_N = \sum_{\{\mathbf{s}\}} e^{-H(\{\mathbf{s}\})/(k_B T)},$$

by deriving a recurrence relation between Z_N and Z_{N+1}. Determine Z_N for the common special case $J_i \equiv J \ \forall i$.

b) Calculate the spin correlation function $\langle s_i s_{i+j} \rangle$ and specialize the result again for $J_i \equiv J \ \forall i$.

c) Show that the spontaneous magnetization $M_s = \langle s \rangle$ (for homogenous interactions $J_i \equiv J$) can only take two values for the infinitely large system:

$$M_s(T) = \begin{cases} 0\,, & T > 0 \\ 1\,, & T = 0 \end{cases} .$$

Make use of $\langle s_i s_{i+j} \rangle \stackrel{j\to\infty}{\longrightarrow} \langle s \rangle^2$.

14. **2-dimensional Ising model** **(5 points)**

The Hamiltonian of the Ising model on an arbitrary regular lattice reads

$$H = -J \sum_{\langle i,j \rangle} s_i s_j \, ,$$

where $s_i = \pm 1$ and $J > 0$ (ferromagnetic coupling). The sum runs over all q pairs of spins that are next neighbors on the lattice. The magnetization is defined as $M = \langle s \rangle$.

a) Determine the energy of the ith spin using the mean field approximation and compute the corresponding partition function.

b) Show that the self-consistency equation

$$M_s = \frac{k_\mathrm{B} T}{qJ} \operatorname{arctanh}(M_s)$$

holds for the mean field approximation. Use it to analyze the critical behavior of the magnet. Calculate the critical temperature for a square lattice and compare your result with the Onsager's solution ($T_c \approx 2.27\, J/k_\mathrm{B}$).

Hint: For $T \approx T_c$ you can assume $M_s \ll 1$.

Total score: 9 points