Statistical Physics II
Problem Set 8

Due: Tuesday, June 3, **before** the lecture

11. **Landau theory** **(7 points)**

Following Landau, assume that the relevant free energy $\mathcal{L}$ may at a critical point generally be expanded in a Taylor series,

$$\mathcal{L} = at\psi^2 + b\psi^4 - h\psi \qquad (a, b > 0,\ t \equiv (T - T_C)/T_C \to 0)\,.$$

The order parameter ψ describes the new quality occuring at the phase transition (e.g. spontaneous magnetization, Higgs field, ...).

a) Sketch $\mathcal{L}$ as a function of ψ for $t < 0$, $t = 0$ and $t > 0$ at $h = 0$ and $h \neq 0$. Identify the thermodynamically stable free energy in each case.

b) Show that $\psi \sim (-t)^\beta$ for $h = 0, t < 0$ and determine β.

c) Show that the critical isotherm locally takes the form $h \sim \psi^\delta$ and determine δ.

d) Discuss the divergence of the isothermal susceptibility $\chi_T = (\partial\psi/\partial h)_T$ (exponents and amplitudes for $t \gtrless 0$).

12.* **Correlation function** **(6 additional points)**

Consider the Landau-Ginzburg functional,

$$L_G = n_c k_B T_c \int_V \mathrm{d}\mathbf{r}\, \mathcal{L}_G\,, \qquad \mathcal{L}_G = \frac{\ell^2}{2}(\nabla\psi)^2 + \frac{t}{2}\psi^2 + \frac{g}{4}\psi^4\,.$$

a) To quadratic leading order show that $\langle|\psi_\mathbf{q}|^2\rangle = h_\mathbf{q}$ satisfies the equation $(q^2 + \xi^{-2})h_\mathbf{q} = c$, where $c = T/(T_c n_c \ell^2)$ and $\xi = \ell|t|^{-1/2}$ is the correlation length of the order parameter fluctuations. This can be achieved as follows:

i) Argue that the complex Fourier-components $\psi_{\mathbf{q}}$ are not completely independent of each other. Demonstrate this by comparing the real and imaginary parts of $\psi_{\mathbf{q}}$ and $\psi_{-\mathbf{q}}$.

ii) Compute the mean value

$$\langle|\psi_{\mathbf{q}}|^2\rangle = \frac{\int \mathrm{d}\psi_{\mathbf{q}}\, |\psi_{\mathbf{q}}|^2 \mathrm{e}^{-L_G/k_B T}}{\int \mathrm{d}\psi_{\mathbf{q}}\, \mathrm{e}^{-L_G/k_B T}} \, .$$

By doing so you prove the equipartition theorem.
Hint: Use the identity

$$\delta(\mathbf{q}) = \frac{1}{V}\int_V \mathrm{d}\mathbf{r} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \, .$$

b) Use a) to show that $h_{\mathbf{q}}$ satisfies the partial differential equation $\nabla^2 h - \xi^{-2} h = -c\delta(\mathbf{r})$.

c) Use b) to show that $h(r)$ with $r \equiv |\mathbf{r}| > 0$ satisfies the differential equation

$$\frac{\mathrm{d}^2 h}{\mathrm{d}r^2} + \frac{(d-1)}{r}\frac{\mathrm{d}h}{\mathrm{d}r} - \frac{h}{\xi^2} = 0\, ,$$

where d is the dimension of the system.

d) Solve the differential equation given in c) for the cases $\xi \to \infty$ and $r \to \infty$.
Hint: Use the ansatz $h(r) = r^{-\alpha} e^{-r/\xi}$.

Total score: 7 points + 6 additional points