Prof. Dr. K. Kroy SS 2014

Statistical Physics II
Problem Set 14

Due: Friday, Juli 18, **before** the seminar

21*. **The Wiener-Khinchin theorem** **(2 additional points)**

Let $Z(t)$ be a stationary process. Show that the relation

$$\langle Z(\omega) Z^*(\omega')\rangle = \sqrt{2\pi} S(\omega)\delta(\omega - \omega')$$

holds, where $S(\omega)$ is the power spectral density in the frequency range $[\omega, \omega + \mathrm{d}\omega]$, defined as the Fourier transform of the autocorrelation function $\langle Z(t) Z^*(0)\rangle$.
Hint: Your prefactor might differ from $\sqrt{2\pi}$, depending on your choice of Fourier coefficients.

22*. **Kramers-Kronig relations** **(10 additional points)**

a) Show for a causal response function of the form

$$\chi(t) = 2i\chi''(t)\Theta(t),$$

that

$$\chi(z) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{izt}\, \chi(t), \; (\mathrm{Im}\, z > 0) \quad \text{and} \quad \chi''_\omega \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{i\omega t}\, \chi''(t)$$

are related to

$$\chi'_\omega = ⨏_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\pi} \frac{\chi''(\omega')}{\omega' - \omega}\,, \tag{1}$$

where $\chi_\omega = \lim_{\epsilon\to 0} \chi(\omega + i\epsilon) = \chi'_\omega + i\chi''_\omega$ and $⨏$ is the principal part integral.
Hint: Use the Dirac identity.

b) Show that causality implies that $\chi(z)$ is analytic in the upper half of the complex z plane.
Hint: Use the Cauchy-Riemann equations.

c) Use the Cauchy integral formula for $\chi(z)$ to show that analogously

$$\chi''_\omega = -⨍_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\pi} \frac{\chi'(\omega')}{\omega' - \omega} \tag{2}$$

holds.

d) Show that χ'_ω is an even and χ''_ω an odd function.

e) Show that the Kramers-Kronig relation for χ''_ω can be written as

$$\chi''_\omega = -\frac{2}{\pi} ⨍_{0}^{\infty} \mathrm{d}\omega' \, \frac{\chi'(\omega')}{\omega'^2 - \omega^2}.$$

Hint: Use that χ'_ω is even.

Total score: 12 additional points