Prof. Dr. K. Kroy SS 2014

Statistical Physics II
Problem Set 13

Due: Tuesday, Juli 8, **before** the lecture

19. **Harmonic oscillator (part II)** **(3+3* points)**

Again, consider the following equation of motion,

$$m\ddot{x}(t) + m\omega_0^2 x(t) + \zeta\dot{x}(t) = \xi(t)\,.$$

a*) Show for the external force $\xi(t) = f_0\cos(\omega t)$ that

$$x(t) = f_0|\chi(\omega)|\cos\big(\omega t - \phi(\omega)\big)$$

is a stationary solution of the oscillator equation, where the phase shift ϕ is determined by $\tan\big(\phi(\omega)\big) = \chi_{\mathrm{im}}(\omega)/\chi_{\mathrm{re}}(\omega)$. Show that the average power $P = \int_0^T \mathrm{d}W(t)/T$ dissipated during one period $T = 2\pi/\omega$ is proportional to $\omega\chi_{\mathrm{im}}(\omega)$, where $W(t)$ is the work done by the external force. What is the meaning of the positivity of $\omega\chi_{\mathrm{im}}(\omega)$ in thermodynamic equilibrium?

b) Calculate the correlation function $C_{xx}(\omega)$ for the case that $\xi(t)$ is a random force with white noise ($\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = 2\zeta k_B T\delta(t-t')$). Establish the relation between your result and the function $\chi_{\mathrm{im}}(\omega)$, determined in 17b). What is the physical interpretation of this relation?

20. **Langevin equation** **(8 + 2* points)**

Consider the 1-dimensional Langevin equation

$$m\dot{v}(t) + \zeta v(t) = \xi(t)$$

as a model for a Brownian particle, where $\xi(t)$ is a Gaussian random force, for which $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = 2\zeta k_B T\delta(t-t')$ holds.

a) Determine $v(t)$ for the initial condition $v(t = 0) = v_0$. Calculate the conditional average values $\langle v(t)\rangle_{v_0}$ and $\langle v(t)v(t')\rangle_{v_0}$, and sketch them as a function of t.

b) Show that the steady state is reached for long times $t, t' \gg m/\zeta$, that is, the initial condition is „forgotten", the velocity correlation function becomes time-translation invariant, and the equipartition theorem is recovered.

c) Calculate the mean position $\langle x(t)\rangle_{v_0 x_0}$ of a Brownian particle, that starts with the velocity $v(t = 0) = v_0$ at $x(t = 0) = x_0$, and the mean-square displacement $\Delta x(t, t') \equiv \langle [x(t) - x(t')]^2\rangle_{v_0}$ for $t - t' > 0$. Determine the variance $\Delta_x(t) \equiv \langle [x(t) - \langle x(t)\rangle]^2\rangle$ by reducing it to known quantities.

d*) Analyse the long-time behavior ($t, t' \gg m/\zeta$) of $\Delta x(t, t')$ and distinguish the cases $|t - t'| \gg m/\zeta$ and $|t - t'| \ll m/\zeta$. Show that the diffusion coefficient defined via a reasonable limit fulfills the Einstein relation.

Total score: 11 points + 5 additional points