Statistical Physics II
Problem Set 10

Due: Tuesday, June 17, **before** the lecture

15. **Renormalization group transformation (RGT) part I (10 points)**

Consider the Ising model in 2 dimensions on a square lattice. The partition sum for N Ising spins s_i ($i = 1, \ldots, N$) is given by

$$Z_N(K_0) = \sum_{\{s_i\}} \exp\left(K_0 \sum_{\langle i,j\rangle} s_i s_j\right) ,$$

where $K_0 = \beta J$ is the coupling constant and $\langle i, j\rangle$ symbolizes the sum over nearest neighbors.

a) Next-nearest neighbors form two diagonal sublattices. In a first renormalization step perform a spin decimation. Consider a spin s in one sublattice and its nearest neighbors s_1, s_2, s_3, s_4 on the other sublattice. Show that

$$\sum_{s=\pm 1} e^{K_0 s(s_1+s_2+s_3+s_4)} = A \exp\left(K' \sum_{1\le i<j\le 4} s_i s_j + U s_1 s_2 s_3 s_4\right)$$

holds and determine A, K' and U.
Taking the partition sum over all spins $s = \pm 1$ of the first sublattice halves the number of degrees of freedom, and a renormalized Hamiltonian can be defined, which contains additional interactions. The original vector of the coupling constants $\mathbf{K}_0 = (K_0, 0, 0, 0, \ldots)$ is transformed to $\mathbf{K}_1 = (K_1, L_1, U_1, 0, \ldots)$ with the new nearest neighbor interaction K_1, next-nearest neighbor interaction L_1, and four-spin coupling U_1. Establish the relation between K_1, L_1, U_1 and K', U, K_0.

b) Since a systematic implementation of further RGT steps is very complex, assume in the following that $K_0 \ll 1$ and consider only the leading order of K_n. Determine the RG–equations $K_1(K_0)$ and $L_1(K_0)$, and the four-spin coupling constant U_1 for the first renormalization step.

c) Consider an extended model with an additional interaction of next-nearest neighbors with coupling $L_0 \ll 1$. How are the RG–equations from b) modified in the first renormalization step? Take these equations as a starting point for the explicit approximative formulation of the RG–equations $K_{n+1}(K_n)$ and $L_{n+1}(L_n)$. Determine the non-trivial fixed point (K^*, L^*).

Total score: 10 points