Universität Leipzig, Institut für Theoretische Physik Prof. Dr. Gerd Rudolph

Exercises in Advanced Quantum Mechanics

Due Thursday, January 29, 2015

34. We consider a hydrogen atom in a weak homogeneous time-periodic electric field of strength $\vec{E} = \vec{E}_0 \cos \omega t$ with $\hbar\omega$ larger than the binding energy. Assume that the system is contained in a cubic box of side length L and that emitted electrons are described by standing plane waves $\psi_{\vec{k}}$ with wave vector $\vec{k}$ in this box.

a) Show that the density of states for the emitted electrons per energy and per solid angle is given by

$$\rho(E) = 2\left(\frac{L}{2\pi}\right)^3 \frac{mk}{\hbar^2}\,.$$

b) Show that, up to a constant energy shift, the Hamiltonian is given by $H = H_0 + H_1$ with

$$H_0 = \frac{\vec{p}^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}\,, \qquad H_1 = V\mathrm{e}^{\mathrm{i}\omega t} + V^\dagger \mathrm{e}^{-\mathrm{i}\omega t}\,,$$

where

$$V = \frac{e}{2\mathrm{i}m\omega}\,\vec{E}_0 \cdot \vec{p}\,.$$

Hint. Start from $H = \frac{1}{2m}\big(\vec{p} - \frac{e}{m}\vec{A}\big)^2 - \frac{e^2}{4\pi\varepsilon_0 r}$ with $\vec{A}$ determined from $\vec{E}$.

c) Determine the transition probability per time and per solid angle from the ground state to the state $\psi_{\vec{k}}$ in first order perturbation theory by means of Fermi's Golden Rule.

35. (Mandatory) Consider the one-particle Hilbert space $\mathcal{H}$ of the positive energy solutions of the free Klein-Gordon equation, given in the momentum representation by $L^2(\Gamma_+, \frac{\mathrm{d}^3k}{2k^0})$.

a) Show that in position representation, the scalar product on $\mathcal{H}$ is given by

$$(\phi, \psi)_\Sigma = \mathrm{i}\int_\Sigma d^3x\,\big(\bar{\phi}(x)\partial_0\psi(x) - \partial_0\bar{\phi}(x)\psi(x)\big)\,, \qquad \phi, \psi \in \mathcal{H}\,,$$

where Σ denotes the hyperplane in Minkowski space defined by $x^0 = 0$.

b) Show that the representation of the proper orthochronous Lorentz group on $\mathcal{H}$ defined by

$$\big(D(\Lambda)\phi\big)(x) := \phi(\Lambda^{-1}x)\,, \qquad \Lambda \in L_+^\uparrow$$

is unitary.