Exercises in Advanced Quantum Mechanics

Due Thursday, December 18, 2014

22. (Mandatory) We consider the harmonic oscillator in one dimension under the influence of a weak constant force F:

$$\hat{H} = \hat{H}_0 + \hat{H}_1\,, \qquad \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2\,, \qquad \hat{H}_1 = -F\hat{x}\,.$$

a) Calculate the second order perturbative corrections to the n-th energy eigenvalue and eigenvector.

b) Determine the exact energy eigenvalues of $\hat{H}$ und compare with the result of a.

c) (1 bonus point) Expand the exact ground state with respect to the eigenvectors of H_0 and compare with the perturbatively corrected ground state of a).

23. Calculate the first order perturbative correction to the ground state energy of a hydrogen atom caused by the finite size of the nucleus. For that purpose, model the nucleus by a homogeneously charged ball of radius R.

24. A reasonable approximation for the Hamiltonian of an alcali atom is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \quad \text{with} \quad \hat{H}_0 = \frac{\hat{\vec{p}}^2}{2} - \frac{e^2}{r} \quad \text{and} \quad \hat{H}_1 = -c_0\frac{e^2}{r} - c_1\frac{e^2}{r^2} - c_2\frac{e^2}{r^3} - c_3\frac{e^2}{r^4}\,,$$

where, for simplicity, $\hbar$ and the electron mass m_e are set equal to 1.

a) Prove the following recursion formula, where $\nu \in \mathbf{Z}$:

$$-\nu\frac{a^2}{n^2}\langle nlm|\frac{1}{r^{\nu+1}}|nlm\rangle + (2\nu+1)a\langle nlm|\frac{1}{r^{\nu+2}}|nlm\rangle + (\nu+1)\left(\frac{\nu(\nu+2)}{4} - l(l+1)\right)\langle nlm|\frac{1}{r^{\nu+3}}|nlm\rangle = 0\,.$$

Hint. Find a linear combination of the commutators $[\frac{1}{r^\nu}[r,\hat{H}_0],\hat{H}_0]$ and $\left[\frac{1}{r^{\nu+1}},\hat{H}_0\right]$ which can be expressed in terms of $\hat{H}_0$, $\hat{\vec{L}}^2$ and powers of $\frac{1}{r}$.

b) Using a), calculate the first order perturbative correction to the energy eigenvalues.

Hint. $\langle nlm|r^{-2}|nlm\rangle = \dfrac{a^2}{n^3(l+\frac{1}{2})}$.