Universität Leipzig, Institut für Theoretische Physik Prof. Dr. Gerd Rudolph

Exercises in Advanced Quantum Mechanics

Due Thursday, December 11, 2014

19. Show that in the position representation, the annihilation operator $\hat{a}(\xi)$ for bosons is given by the formula

$$\hat{a}(\xi)|\psi_N\rangle = \frac{\sqrt{N}}{\sqrt{(N-1)!}} \int \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1} \;\; \psi_N(\xi_1, \ldots, \xi_{N-1}, \xi) \;\; \hat{a}^\dagger(\xi_{N-1}) \cdots \hat{a}^\dagger(\xi_1) \;\; |0\rangle \, .$$

Derive an analogous formula for the creation operator $\hat{a}^\dagger(\xi)$.

20. (Mandatory) We consider a quantum system in a cubic volume V of side length L, described by the Hamiltonian

$$H = \sum\nolimits_{\vec{k}\neq 0} \left(\frac{\vec{k}^{\,2}}{2m} + nU_{\vec{k}} \right) a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{n}{2} \sum\nolimits_{\vec{k}\neq 0} U_{\vec{k}} \left(a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} + a_{\vec{k}} a_{-\vec{k}} \right) + \frac{n^2 V U_0}{2} \, ,$$

where the sum runs over wave vectors $\vec{k} = \frac{2\pi}{L}(m_x, m_y, m_z)$ with m_x, m_y, m_z being integers, n is the particle density, $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ are the annihilation and creation operators of a particle in the momentum eigenstate $\vec{k}$ and $U_{\vec{k}}$ are the Fourier coefficients of the potential. We carry out a Bogolyubov transformation $\hat{a}_{\vec{k}} \mapsto \hat{A}_{\vec{k}}$, where

$$\hat{a}_{\vec{k}} = u_{\vec{k}} \, \hat{A}_{\vec{k}} + v_{\vec{k}} \, \hat{A}^\dagger_{-\vec{k}} \, , \qquad \hat{a}^\dagger_{\vec{k}} = u_{\vec{k}} \, \hat{A}^\dagger_{\vec{k}} + v_{\vec{k}} \, \hat{A}_{-\vec{k}} \, ,$$

with $u_{\vec{k}}, v_{\vec{k}} \in \mathbb{R}$ such that $u_{\vec{k}} = u_{-\vec{k}}$ and $v_{\vec{k}} = v_{-\vec{k}}$.

a) Show that the canonical commutation relations

$$[\hat{A}_{\vec{k}}, \hat{A}^\dagger_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'} \, , \qquad [\hat{A}_{\vec{k}}, \hat{A}_{\vec{k}'}] = 0$$

are satisfied if and only if

$$u^2_{\vec{k}} - v^2_{\vec{k}} = 1 \, . \tag{1}$$

b) Show that after the Bogolyubov transformation the Hamiltonian is diagonal if and only if

$$\left(\frac{\vec{k}^{\,2}}{2m} + nU_{\vec{k}} \right) u_{\vec{k}} \, v_{\vec{k}} + \frac{n}{2} \, U_{\vec{k}} \, (u^2_{\vec{k}} + v^2_{\vec{k}}) = 0 \, . \tag{2}$$

21. For the Bogolyubov transformation of Problem 20, determine the expressions $u_{\vec{k}}^2$, $v_{\vec{k}}^2$ and $u_{\vec{k}}v_{\vec{k}}$ from the relations (1) and (2) and show that the Hamiltonian takes the form

$$H = \sum_{\vec{k}\neq 0} \omega_{\vec{k}} A_{\vec{k}}^\dagger A_{\vec{k}} + \frac{1}{2}\sum_{\vec{k}\neq 0}\left(\omega_{\vec{k}} - \frac{\vec{k}^{\,2}}{2m} - nU_{\vec{k}}\right) + \frac{1}{2}n^2 V U_0\,,$$

where

$$\omega_{\vec{k}} = \sqrt{\left(\frac{\vec{k}^{\,2}}{2m}\right)^2 + \frac{\vec{k}^{\,2}}{m} nU_{\vec{k}}}\,.$$