Exercises in Advanced Quantum Mechanics

Due Thursday, November 27, 2014

14. (Mandatory) We consider a charged particle in a spherically symmetric potential V and a homogeneous electric field of magnitude E in z-direction. The Hamiltonian is given by

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V(r) - qzE\,.$$

a) Determine the symmetry group of the system and show that there exists a common eigenbasis of $\hat{H}$ and $\hat{L}_3$.

b) Show that

$$[\hat{L}_\pm, H] = \pm\hbar q E \hat{x}_\pm$$

where $\hat{L}_\pm = \hat{L}_1 \pm \mathrm{i}\hat{L}_2$ and $\hat{x}_\pm = \hat{x}_1 \pm \mathrm{i}\hat{x}_2$. Use this to argue why the electric field breaks the degeneracy of the energy levels with respect to the quantum number m of $\hat{L}_z$ which is present in the case $E = 0$ (Stark effect).

15. In quantum mechanics, for an autonomous system, the analogue of a classical Galilei transformation

$$\vec{x}\,' = \vec{x} - t\vec{v}\,, \qquad t' = t$$

is defined by

$$\psi'(\vec{x}, t) = \big(\hat{G}(\vec{v}, t)\psi\big)(\vec{x}, t)$$

with

$$\hat{G}(\vec{v}, t) = e^{-\frac{i}{\hbar}\hat{H}t} \;\; \hat{G}(\vec{v}) \;\; e^{\frac{i}{\hbar}\hat{H}t}\,, \qquad \hat{G}(\vec{v})\,|\vec{p}\rangle := |\vec{p} - m\vec{v}\rangle\,.$$

Using results from Exercise 9, show that for a spinless free particle, the following holds.

a) $\psi'(\vec{x}, t) = e^{-\frac{i}{\hbar}m\vec{v}\left(\vec{x}+\frac{1}{2}\vec{v}t\right)} \;\psi(\vec{x} + \vec{v}t, t)\,.$

b) The Galilei transformation leaves invariant the Schrödinger equation, i.e.,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t) = -\frac{\hbar^2}{2m}\Delta\psi(\vec{x}, t)$$

implies

$$\mathrm{i}\hbar\frac{\partial}{\partial t'}\psi'(\vec{x}\,', t') = -\frac{\hbar^2}{2m}\Delta'\psi'(\vec{x}\,', t')\,.$$

16. For a system of N identical particles, the operators of symmetrization and antisymmetrization are defined by

$$\hat{\mathcal{S}} := \frac{1}{N!} \sum\nolimits_{\pi \in S_N} \hat{\pi}\,, \qquad \hat{\mathcal{A}} := \frac{1}{N!} \sum\nolimits_{\pi \in S_N} \epsilon(\pi)\, \hat{\pi}\,,$$

where S_N denotes the group of permutations of N elements and $\epsilon(\pi)$ denotes the sign of the permutation π, i.e.,

$$\epsilon(\pi) = \begin{cases} 1 & \pi \text{ is even,} \\ -1 & \pi \text{ is odd.} \end{cases}$$

Show that $\hat{\mathcal{S}}$ and $\hat{\mathcal{A}}$ are orthoprojectors and that they satisfy $\hat{\mathcal{S}}\,\hat{\mathcal{A}} = \hat{\mathcal{A}}\,\hat{\mathcal{S}} = 0$.