Universität Leipzig, Institut für Theoretische Physik Prof. Dr. Gerd Rudolph

Exercises in Advanced Quantum Mechanics

Due Thursday, November 20, 2014

11. (Mandatory) We consider the Hilbert space $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of two particles of spin $\frac{1}{2}$. The operator of the dipole moment is given by

$$\hat{\vec{d}} := e\left(\hat{\vec{x}} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\vec{x}}\right) \otimes \left(\mathbb{1}_2 \otimes \mathbb{1}_2\right).$$

a) Use the infinitesimal criterion to show that $\hat{\vec{d}}$ is a vector operator with respect to both orbital angular momentum $\hat{\vec{L}}$ and total angular momentum $\hat{\vec{J}}$.

b) Determine the irreducible tensor operator corresponding to $\hat{\vec{d}}$ and use the Wigner-Eckart theorem to prove the following selection rules for dipole radiation. The matrix elements $\langle \alpha'\, j'\, m_j' |\, \hat{\vec{d}} | \alpha\, j\, m_j \rangle$ vanish unless

$$|j' - j| \leq 1\,, \qquad |m_j' - m_j| \leq 1\,.$$

and the matrix elements $\langle \alpha'\, l'\, m_l' |\, \hat{\vec{d}} | \alpha\, l\, m_l \rangle$ vanish unless

$$|l' - l| \leq 1\,, \qquad |m_l' - m_l| \leq 1\,.$$

c) (optional) Can the selection rule for l be sharpened by means of the parity operator?

12. We consider a representation of the group of rotations on the Hilbert space of a quantum mechanical system. Let $\hat{\vec{J}}$ be the corresponding operator of angular momentum and let $\{|\alpha\, j\, m\rangle\}$ be an orthonormal basis of common eigenvectors of $\hat{\vec{J}}^2$ and $\hat{J}_3$, where α refers to the quantum numbers of certain further observables which complement $\hat{\vec{J}}^2$ and $\hat{J}_3$ to a complete system of commuting observables. Let $\hat{\vec{K}}$ be a vector operator with respect to this representation. Use the Wigner-Eckart theorem to show that the matrix elements of the components $\hat{K}_i$ of $\hat{\vec{K}}$ can be expressed in terms of the matrix elements of $\hat{J}_i$ and of the projection $\hat{\vec{K}} \cdot \hat{\vec{J}}$ as follows:

$$\langle \alpha\, j\, m | \hat{K}_i | \alpha\, j\, m' \rangle = \tfrac{1}{\hbar^2 j(j+1)}\, \langle \alpha\, j\, m | \hat{J}_i | \alpha\, j\, m' \rangle\, \langle \alpha\, j\, m | \hat{\vec{K}} \cdot \hat{\vec{J}} | \alpha\, j\, m \rangle\,.$$

Hint. Start with showing that $\langle\alpha\, j\, m|\hat{K}_i|\alpha\, j\, m'\rangle = \langle\alpha\, j\, m|\hat{J}_i|\alpha\, j\, m'\rangle\, \frac{\langle\alpha j\|\hat{K}^1\|\alpha j\rangle}{\langle\alpha j\|\hat{J}^1\|\alpha j\rangle}$, where $\langle\alpha j\|\hat{K}^1\|\alpha j\rangle$ and $\langle\alpha j\|\hat{J}^1\|\alpha j\rangle$ denote the reduced matrix elements of the irreducible tensor operators $\hat{K}^1_M$ and $\hat{J}^1_M$ associated with $\hat{\vec{K}}$ and $\hat{\vec{J}}$, respectively. Then, check that $\hat{\vec{K}} \cdot \hat{\vec{J}} = \sum_{M=-1}^{1}\ (-1)^M\ \hat{K}^1_M\ \hat{J}^1_{-M}$ and use this to rewrite the reduced matrix elements $\langle\alpha j\|\hat{K}^1\|\alpha j\rangle$ and $\langle\alpha j\|\hat{J}^1\|\alpha j\rangle$ in terms of the matrix elements $\langle\alpha\, j\, m|\hat{\vec{K}} \cdot \hat{\vec{J}}|\alpha\, j\, m\rangle$ and $\langle\alpha\, j\, m|\hat{\vec{J}}^2|\alpha\, j\, m\rangle$, respectively.

13. Prove the Kramers Rule: if the Hamiltonian $\hat{H}$ of a system of particles with spin $\frac{1}{2}$ is invariant under time reversal and if the number of particles is odd, then all eigenvalues of $\hat{H}$ have even multiplicity.