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Exercises in Advanced Quantum Mechanics

Due Thursday, October 30, 2014

3. (Mandatory) We consider a system of two particles of spin $\frac{1}{2}$, each described by the two-dimensional one-particle Hilbert space $\mathcal{H}$. Let $|\pm\rangle \in \mathcal{H}$ denote the eigenvectors of the spin operator $\hat{S}_3$.

a) Show that the vectors

$$\varphi^{\pm} = \tfrac{1}{\sqrt{2}}\big(\,|+\rangle \otimes |+\rangle \pm |-\rangle \otimes |-\rangle\,\big)\,, \qquad \psi^{\pm} = \tfrac{1}{\sqrt{2}}\big(\,|+\rangle \otimes |-\rangle \pm |-\rangle \otimes |+\rangle\,\big) \tag{1}$$

form an orthonormal basis of maximally entangled vectors (Bell basis) in $\mathcal{H} \otimes \mathcal{H}$.

b) Find out how the spin operators $\hat{\vec{S}} \otimes \mathbb{1}$ and $\mathbb{1} \otimes \hat{\vec{S}}$ act on these vectors.

c) Show that $\hat{S}_1 \otimes \hat{S}_1$ and $\hat{S}_3 \otimes \hat{S}_3$ commute. Give a physical interpretation of these two observables.

4. Let $\hat{A}_i$ and $\hat{B}_i$, $i = 1, 2$, be observables having pure eigenvalue spectrum with eigenvalues between -1 and 1 and satisfying $[A_i, B_j] = 0$ for $i, j = 1, 2$.

a) Show that in every state $\hat{\rho}$, the Bell correlation has the upper bound

$$\mathrm{tr}\left(\hat{\rho}\left(\hat{A}_1(\hat{B}_1 + \hat{B}_2) + \hat{A}_2(\hat{B}_1 - \hat{B}_2)\right)\right) \leq 2\sqrt{2}\,. \tag{2}$$

Hint. Show that the operators

$$\hat{A} = \tfrac{1}{2}(\hat{A}_1 + i\hat{A}_2)\,, \qquad \hat{B} = \tfrac{1}{2\sqrt{2}}\left((\hat{B}_1 + \hat{B}_2) + i(\hat{B}_1 - \hat{B}_2)\right)$$

satisfy $\hat{A}^\dagger\hat{A} + \hat{A}\hat{A}^\dagger \leq \mathbf{1}$ and $\hat{B}^\dagger\hat{B} + \hat{B}\hat{B}^\dagger \leq \mathbf{1}$. Use this and the obvious inequality $(A - B)^\dagger(A - B) + (A - B)(A - B)^\dagger \geq 0$ to prove

$$\hat{A}_1(\hat{B}_1 + \hat{B}_2) + \hat{A}_2(\hat{B}_1 - \hat{B}_2) \leq 2\sqrt{2}\,\mathbb{1}\,.$$

b) Consider a system of two particles of spin $\frac{1}{2}$. Let

$$\hat{A}_i = \sigma_i \otimes \mathbb{1}\,, \qquad \hat{B}_1 = \tfrac{1}{\sqrt{2}}\mathbb{1} \otimes (\sigma_1 + \sigma_2)\,, \qquad \hat{B}_2 = \tfrac{1}{\sqrt{2}}\mathbb{1} \otimes (\sigma_1 - \sigma_2)$$

and let $\hat{\rho}$ be the density operator corresponding to the pure state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(\,|+\rangle \otimes |-\rangle \;+\; |-\rangle \otimes |+\rangle\,\big)\,.$$

Show that in this case one has equality in Formula (2).

5. We consider a system of two particles, described by the Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$. Let $\hat{A}_1^{(1)}$, $\hat{A}_2^{(1)}$ and $\hat{B}_1^{(2)}$, $\hat{B}_2^{(2)}$ be commuting observables on $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, respectively, having pure eigenvalue spectrum with eigenvalues between -1 and 1. Define

$$\hat{A}_i := \hat{A}_i^{(1)} \otimes \mathbb{1}\,, \qquad \hat{B}_i := \mathbb{1} \otimes \hat{B}_i^{(2)}\,, \qquad i = 1, 2\,.$$

Show that for a separable state $\hat{\rho}$ of the coupled system $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, the Bell correlation has the upper bound

$$\mathrm{tr}\left(\hat{\rho}\left(\hat{A}_1(\hat{B}_1 + \hat{B}_2) + \hat{A}_2(\hat{B}_1 - \hat{B}_2)\right)\right) \leq 2\,.$$

(This means that separable states do not violate the Bell inequality.)