Exercises in Advanced Quantum Mechanics

Due Thursday, October 23, 2014

1. Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces, let $\hat{A}$ and $\hat{C}$ be bounded linear operators on $\mathcal{H}$ and let $\hat{B}$, $\hat{D}$ be bounded linear operators on $\mathcal{H}'$. Verify the following formulae.

a) $(\hat{A} \otimes \hat{B})(\hat{C} \otimes \hat{D}) = \hat{A}\hat{C} \otimes \hat{B}\hat{D}$.

b) $(\hat{A} \otimes \hat{B})^\dagger = \hat{A}^\dagger \otimes \hat{B}^\dagger$.

c) If $\hat{A} \geq 0$ and $\hat{B} \geq 0$, then $\hat{A} \otimes \hat{B} \geq 0$

d) If $\hat{A}$ and $\hat{B}$ are bounded, then $\hat{A} \otimes \hat{B}$ is bounded and satisfies $\|\hat{A} \otimes B\| = \|\hat{A}\| \, \|\hat{B}\|$.

e) If $\hat{A}$ and $\hat{B}$ are trace class operators, then $\hat{A} \otimes \hat{B}$ is a trace class operator and one has $\operatorname{tr}\big(\hat{A} \otimes \hat{B}\big) = \operatorname{tr}(\hat{A}) \operatorname{tr}(\hat{B})$.

Hint. A bounded linear operator A on $\mathcal{H}$ is of trace class if the series $\sum_n \langle \varphi_n | A \varphi_n \rangle$ converges absolutely for some orthonormal basis $\{\varphi_n\}$ in $\mathcal{H}$. In this case, the series converges for every orthonormal basis and the limit does not depend on the basis. The trace $\operatorname{tr}(A)$ is defined by this limit.

2. (Mandatory) Consider a system consisting of two particles of spin $\frac{1}{2}$. Let $\hat{S}^{(i)}$, $i = 1, 2$, denote the spin operators of the single particles, given by

$$\hat{S}_x^{(i)} = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{S}_y^{(i)} = \frac{\hbar}{2} \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \hat{S}_z^{(i)} = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and let

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For arbitrary vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, determine the expectation values of the observables

$$\big(2\vec{a} \cdot \hat{\vec{S}}^{(1)}\big) \otimes \mathbb{1}\,, \qquad \mathbb{1} \otimes \big(2\vec{b} \cdot \hat{\vec{S}}^{(2)}\big)\,, \qquad \big(2\vec{a} \cdot \hat{\vec{S}}^{(1)}\big) \otimes \big(2\vec{b} \cdot \hat{\vec{S}}^{(2)}\big)$$

in the singlet state $|\,\phi\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle \otimes |-\rangle \;-\; |-\rangle \otimes |+\rangle \Big)$.