Chapter 10

Summary of time independent electrodynamics

10.1 Electrostatics

- Physical law

  Coulomb's law – charges as origin of electric field
  Superposition principle
  Vector of the electric field $\mathbf{E}(\mathbf{x})$ in vacuum due to charge distribution in volume V given by the volume density of free charges $\rho(\mathbf{x})$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x'$$

  d^3x' volume element ($= dx'dy'dz'$ in Cartesian coordinates)
  Charge density of a discrete set of charged point particles at positions $\mathbf{x}_i$

$$\rho(\mathbf{x}) = \sum_{i=1}^{n} q_i \, \delta(\mathbf{x} - \mathbf{x}_i)$$

  Medium as phenomenological model

- Basic differential and integral equations

  $\mathbf{P}$ – polarization vector, $\mathbf{D}$ – electric displacement vector, $\mathbf{n}$ – outward normal unit vector

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{D}(\mathbf{x}) &= \rho(\mathbf{x}) \,, \qquad & \oint_S \mathbf{D} \cdot \mathbf{n} &= \int_V \rho(\mathbf{x}) \, d^3x = Q^{\text{encl}} \quad \text{(Gauss law)} \\ \boldsymbol{\nabla} \times \mathbf{E}(\mathbf{x}) &= 0 \,, \qquad & \oint_C \mathbf{E} \cdot d\mathbf{l} &= 0 \\ \mathbf{D} &= \varepsilon_0 \, \mathbf{E} + \mathbf{P} \,, \qquad & \mathbf{D} &= \varepsilon \, \mathbf{E} \end{aligned}$$

  Bounded charges or polarization charges

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} \,, \quad \sigma_b = \mathbf{P} \cdot \mathbf{n}$$

- Boundary conditions at interface between two media

  $\mathbf{n}_{21}$ – normal unit vector from medium 1 to medium 2
  σ – free surface charge density

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{21} = \sigma\,, \quad \mathbf{n}_{21} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$$

- Electrostatic scalar potential $\Phi(\mathbf{x})$

$$\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{x}) = 0 \quad \Rightarrow \quad \mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\Phi(\mathbf{x})$$

- Poisson equation for isotropic and linear media (ε position and direction independent) ($\rho = 0$: Laplace equation)

$$\boldsymbol{\nabla}^2\Phi \equiv \Delta\Phi = -\frac{\rho}{\varepsilon}$$

  Solution without bounding surfaces

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'$$

- Solution of the Poisson equation in the presence of bounding surfaces

  Use Green's theorems and Green functions
  Green functions – solution with δ-function as inhomogeneous part
  interpretation: potential due to a unit point charge (in units $4\pi\varepsilon$)

$$\boldsymbol{\nabla}'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\,\delta(\mathbf{x}-\mathbf{x}')$$

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}')\,, \quad \boldsymbol{\nabla}'^2 F(\mathbf{x},\mathbf{x}') = 0 \quad \mathbf{x}' \in V$$

  Use freedom in the definition of G via F to satisfy appropriate bounday conditions
  Dirichlet: specification of potential on closed surface S

$$G_D(\mathbf{x},\mathbf{x}') = 0\,, \quad \mathbf{x}' \in S$$

  Potential for $\mathbf{x} \in V$ using Dirichlet boundary conditions

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon}\int_V \rho(\mathbf{x}')\,G_D(\mathbf{x},\mathbf{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}\,da'\,, \quad \frac{\partial G_D}{\partial n'} \equiv \boldsymbol{\nabla}' G_D \cdot \mathbf{n}'$$

  Neumann: specification of electric field (normal derivative of potential) on S

- Solution of Laplace equation with boundary conditions for problems with symmetries

  Method of image charges: mimic boundary conditions by placing image charges of appropriate magnitude at positions in a region external to region of interest and take into account their potentials

Method of separation of variables: construct an ansatz for the solution
Example: Boundary-value problem with azimuthal symmetry using Legendre polynomials (unknown constants A_l, B_l determined from boundary conditions)

$$\Delta\Phi(r,\theta) = 0 \quad \to \quad \Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + B_l\, r^{-l-1}\right)\, P_l(\cos\theta)$$

- Electrostatic energy for linear media

$$W_e = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\, d^3x = \frac{1}{2}\int \rho\,\Phi\, d^3x\,, \quad w_e = \frac{1}{2}\,\mathbf{E}\cdot\mathbf{D}$$

- Multipole expansion of a localized charge distribution in Cartesian coordinates

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{x} + \frac{\mathbf{p}\cdot\mathbf{x}}{x^3} + \frac{1}{2}\,Q_{ij}\,\frac{x_i x_j}{x^5} + \cdots\right)\,, \quad x = |\mathbf{x}| \\
\mathbf{E}(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\,\mathbf{x}}{x^3} + \frac{3(\mathbf{p}\cdot\mathbf{x})\,\mathbf{x} - x^2\,\mathbf{p}}{x^5} + \cdots\right)
\end{aligned}$$

Q – total charge, $\mathbf{p}$ – electric dipole moment, Q_{ij} – traceless quadrupole moment tensor

$$Q = \int \rho(\mathbf{x})\, d^3x\,, \quad \mathbf{p} = \int \mathbf{x}\,\rho(\mathbf{x})\, d^3x\,, \quad Q_{ij} = \int (3x_i x_j - \mathbf{x}^2\,\delta_{ij})\,\rho(\mathbf{x})\, d^3x$$

10.2 Magnetostatics

- Physical laws

Biot and Savart law – absence of magnetic monopoles (individual magnetic charges)
$\mathbf{B}$ – vector of magnetic induction or magnetic flux density due to some free current distribution with density $\mathbf{J}$ (SI-unit A/m^2)

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'$$

Ampere's law – free currents are origins of magnetic fields, $\mathbf{H}$ – magnetic field vector

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I^{\mathrm{encl}}\,, \quad \text{in vacuum} \quad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$$

- Basic equations (current divergenceless: $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, $\mathbf{M}$ – magnetization)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{x}) = 0\,, &\qquad \int_S \mathbf{B}\cdot\mathbf{n}\, da = 0 \\
\boldsymbol{\nabla}\times\mathbf{H}(\mathbf{x}) = \mathbf{J}(\mathbf{x})\,, &\qquad \oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot\mathbf{n}\, da = I^{\mathrm{encl}} \\
\mathbf{H} = \frac{1}{\mu_0}\,\mathbf{B} - \mathbf{M} &
\end{aligned}$$

Diamagnetic and paramagnetic material: $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$
Ferromagnetic material: Nonlinear relation between $\mathbf{H}$ and $\mathbf{B}$
Effective volume and surface currrent densities (SI-units A/m^2 and A/m)

$$\mathbf{J}_e = \boldsymbol{\nabla} \times \mathbf{M}\,, \quad \boldsymbol{\lambda}_e = \mathbf{M} \times \mathbf{n}$$

- Boundary conditions at interface between two media (usually $\boldsymbol{\lambda} \equiv 0$)

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n}_{21} = 0\,, \quad \mathbf{n}_{21} \times (\mathbf{H}_2 - \mathbf{H}_1) = \boldsymbol{\lambda}$$

- Vector potential $\mathbf{A}(\mathbf{x})$

$\mathbf{A}$ defined up to the gradient of an arbitrary scalar function ψ (gauge freedom)

$$\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{x}) = 0 \quad \Rightarrow \quad \mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})$$

For isotropic and linear media (μ constant) and choosing Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ derive from

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{1}{\mu}\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J}$$

the partial differential equation for $\mathbf{A}$

$$\boldsymbol{\nabla}^2\mathbf{A} = \Delta\mathbf{A} = -\mu\,\mathbf{J}$$

Solution in unbounded space (Coulomb gauge)

$$\mathbf{A}(\mathbf{x}) = \frac{\mu}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,d^3x'\,, \quad \psi = \text{const}$$

- Method of scalar magnetic potential

Use in regions with $\mathbf{J} = 0$ (μ constant)

$$\boldsymbol{\nabla} \times \mathbf{H} = 0 \quad \Rightarrow \quad \mathbf{H} = -\boldsymbol{\nabla}\Phi_m$$

Laplace equation for Φ_m

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \boldsymbol{\nabla}^2\Phi_m = \Delta\Phi_m = 0$$

- Various quantities for a localized current distribution $\mathbf{J}(\mathbf{x})$

Force on $\mathbf{J}$ in an external $\mathbf{B}$

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\,d^3x$$

Total torque

$$\mathbf{N} = \int \mathbf{x} \times (\mathbf{J} \times \mathbf{B})\, d^3x$$

Magnetic moment

$$\mathbf{m} = \frac{1}{2} \int_V \mathbf{x} \times \mathbf{J}(\mathbf{x})\, d^3x$$

Vector potential and magnetic field at large distances (in vacuum)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{x^3}\,, \quad \mathbf{B} = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \mathbf{x})\, \mathbf{x} - x^2\, \mathbf{m}}{x^5}$$

- Magnetostatic energy for linear diamagnetic and paramagnetic media in derivation Faraday's law of induction required

$$W_m = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B}\, d^3x = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A}\, d^3x\,, \quad w_m = \frac{1}{2}\, \mathbf{H} \cdot \mathbf{B}$$