Appendix A

Curvilinear coordinates

A.1 Lamé coefficients

Consider set of equations

$$\xi_i = \xi_i(x_1, x_2, x_3)\,, \quad i = 1, 2, 3$$

where ξ_1, ξ_2, ξ_3 independent, single-valued and continuous
(x_1, x_2, x_3) : coordinates of point P in Cartesian system with radius-vector $\mathbf{x}$
(ξ_1, ξ_2, ξ_3) : coordinates of point P in curvilinear system

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$: base unit vectors in Cartesian coordinates
$\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}, \mathbf{e}_{\xi_3}$: base unit vectors in curvilinear system
we restrict ourselves to orthogonal curvilinear coordinates systems

$$\mathbf{e}_{\xi_i} \cdot \mathbf{e}_{\xi_j} = \delta_{ij}$$

Consider $\mathbf{x} = \mathbf{x}(x_i(\xi_j))$ and take the total differential (summation over double appearing indices understood)

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi_j}\, d\xi_j$$

$\dfrac{\partial \mathbf{x}}{\partial \xi_1}$ partial derivative keeping ξ_2, ξ_3 fixed: vector tangential to coordinate line ξ_1

$$\longrightarrow \quad \frac{\partial \mathbf{x}}{\partial \xi_1} = h_1\, \mathbf{e}_{\xi_1}$$

in general

$$d\mathbf{x} = h_1\, d\xi_1\, \mathbf{e}_{\xi_1} + h_2\, d\xi_2\, \mathbf{e}_{\xi_2} + h_3\, d\xi_3\, \mathbf{e}_{\xi_3}$$

h_i Lamé coefficients
Those coefficients (might) depend on the coordinates ξ_i

$$h_i = h_i(\xi_1, \xi_2, \xi_3)$$

$h_i\, d\xi_i$ (no summation) components of length element along $\mathbf{e}_{\xi_i}$

Without restriction to orthogonal systems the length element $ds = |d\mathbf{x}|$ is defined as

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = dx_j\, dx_j = \frac{\partial x_j}{\partial \xi_l} \frac{\partial x_j}{\partial \xi_m}\, d\xi_l\, d\xi_m = g_{lm}\, d\xi_l\, d\xi_m$$

Coefficients

$$g_{lm} = \frac{\partial x_j}{\partial \xi_l} \frac{\partial x_j}{\partial \xi_m}$$

are called the elements of the metric tensor

For an orthogonal curvilinear system the base vectors $\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}, \mathbf{e}_{\xi_3}$ are mutually perpendicular and form a right-handed system
We get the length element squared

$$ds^2 = h_1^2\, (d\xi_1)^2 + h_2^2\, (d\xi_2)^2 + h_3^2\, (d\xi_3)^2$$

the Lamé coefficients

$$g_{11} = h_1^2\,, \quad g_{22} = h_2^2\,, \quad g_{33} = h_3^2\,, \quad g_{ik} = 0 \text{ for } i \neq k$$

the volume element (elementary rectangular box)

$$d^3x \equiv dV = h_1 h_2 h_3\, d\xi_1 d\xi_2 d\xi_3$$

A.2 Some special orthogonal coordinates

- Cartesian coordinates $(-\infty < x_1 < \infty\,,\ -\infty < x_2 < \infty\,,\ -\infty < x_3 < \infty)$

$$\begin{aligned} &h_1 = h_2 = h_3 = 1 \\ &ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \\ &d^3x = dx_1\, dx_2\, dx_3 \end{aligned}$$

- Cylindrical coordinates $(0 \le \rho < \infty\,,\ 0 \le \varphi < 2\pi\,,\ -\infty < z < \infty)$

$$x_1 = \rho\, \cos\varphi\,,\ x_2 = \rho\, \sin\varphi\,,\ x_3 = z$$

$$\begin{aligned} &h_\rho = h_z = 1\,,\ h_\varphi = \rho \\ &ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 \\ &d^3x = \rho\, d\rho\, d\varphi\, dz \end{aligned}$$

- Spherical coordinates $(0 \le r < \infty\,,\ 0 \le \theta \le \pi\,,\ 0 \le \varphi < 2\pi)$

$$x_1 = r\, \sin\theta\, \cos\varphi\,,\ x_2 = r\, \sin\theta\, \sin\varphi\,,\ x_3 = r\, \cos\theta$$

$$\begin{aligned} &h_r = 1\,,\ h_\theta = r\,,\ h_\varphi = r\, \sin\theta \\ &ds^2 = dr^2 + r^2 d\theta^2 + r^2\, \sin\theta^2\, d\varphi^2 \\ &d^3x = r^2\, dr\, \sin\theta\, d\theta\, d\varphi = r^2\, dr\, d\cos\theta\, d\varphi \equiv r^2 dr\, d\Omega \end{aligned}$$

- Parabolic cylindrical coordinates
  parametrization in Mathematica $(0 \le u < \infty\,,\ 0 \le v < \infty\,,\ -\infty < z < \infty)$

$$x_1 = \frac{1}{2}(u^2 - v^2)\,,\ x_2 = u\, v\,,\ x_3 = z$$

$$\begin{aligned} &h_u = h_v = \sqrt{u^2 + v^2}\,,\ h_z = 1 \\ &ds^2 = (u^2 + v^2)\,(du^2 + dv^2) + dz^2 \\ &d^3x = (u^2 + v^2)\, du\, dv\, dz \end{aligned}$$

  parametrization of Arfken $(0 \le \xi < \infty\,,\ 0 \le \eta < \infty\,,\ -\infty < z < \infty)$

$$x_1 \leftrightarrow x_2\,,\ u \to \eta\,,\ v \to \xi$$

- Parabolic coordinates
  parametrization of Arfken, Landau/Lifshitz $(0 \le \xi < \infty\,,\ 0 \le \eta < \infty\,,\ 0 \le \varphi < 2\pi)$

$$x_1 = \sqrt{\xi\,\eta}\, \cos\varphi\,,\ x_2 = \sqrt{\xi\,\eta}\, \sin\varphi\,,\ x_3 = \frac{1}{2}(\xi - \eta)$$

$$
\begin{aligned}
h_\xi &= \frac{1}{2}\sqrt{\frac{\xi+\eta}{\xi}}\,,\ h_\eta = \frac{1}{2}\sqrt{\frac{\xi+\eta}{\eta}}\,,\ h_\varphi = \sqrt{\xi\,\eta} \\
ds^2 &= \frac{1}{4}\frac{\xi+\eta}{\xi}\,d\xi^2 + \frac{1}{4}\frac{\xi+\eta}{\eta}\,d\eta^2 + \xi\eta\,d\varphi^2 \\
d^3x &= \frac{1}{4}\,(\xi+\eta)\,d\xi\,d\eta\,d\varphi
\end{aligned}
$$

parametrization in Mathematica ($0 \le u < \infty\,,\ 0 \le v < \infty\,,\ 0 \le \varphi < 2\pi$)

$$
\xi = u^2\,,\ \eta = v^2\,,\quad x_1 = u\,v\,\cos\varphi\,,\ x_2 = u\,v\sin\varphi\,,\ x_3 = \frac{1}{2}(u^2 - v^2)
$$

$$
\begin{aligned}
h_u &= h_v = \sqrt{u^2+v^2}\,,\ h_\varphi = uv \\
ds^2 &= (u^2+v^2)\,(du^2+dv^2) + uv\,d\varphi^2 \\
d^3x &= uv(u^2+v^2)\,du\,dv\,d\varphi
\end{aligned}
$$

A.3 Vector operations in orthogonal coordinates

Gradient

The length element along coordinate line ξ_1 is $h_1\, d\xi_1$

$$\longrightarrow \quad \mathbf{e}_{\xi_1} \cdot \text{grad}\psi = \mathbf{e}_{\xi_1} \cdot \boldsymbol{\nabla}\psi = \frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1}$$

$$\boxed{\boldsymbol{\nabla}\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial\xi_1}\mathbf{e}_{\xi_1} + \frac{1}{h_2}\frac{\partial\psi}{\partial\xi_2}\mathbf{e}_{\xi_2} + \frac{1}{h_3}\frac{\partial\psi}{\partial\xi_3}\mathbf{e}_{\xi_3}}$$

$\Rightarrow$ form of the nabla operator

$$\boxed{\boldsymbol{\nabla} = \sum_{i=1}^{3} \mathbf{e}_{\xi_i} \frac{1}{h_i}\frac{\partial}{\partial \xi_i}}$$

Divergence and Laplacian

Use the divergence definition

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \mathbf{A} \cdot \mathbf{n}\, da$$

with

$$\mathbf{A} = A_1(\xi_1, \xi_2, \xi_3)\,\mathbf{e}_{\xi_1} + A_2(\xi_1, \xi_2, \xi_2)\,\mathbf{e}_{\xi_1} + A_3(\xi_1, \xi_2, \xi_3)\,\mathbf{e}_{\xi_3}$$

and choose as volume ΔV an elementary rectangular box of sides $h_1\Delta\xi_1 h_2\Delta\xi_2 h_3\Delta\xi_3$ Analogously to the derivation in Cartesian coordinates (compare Chapter 1.2) we calculate the net outward flux in direction of coordinate line ξ_1 divided by the volume:

$$\lim_{\Delta V\to 0} \big[A_1(\xi_1 + \Delta\xi_1, \overline{\xi_2}, \overline{\xi_3})\, h_2(\xi_1 + \Delta\xi_1, \overline{\xi_2}, \overline{\xi_3})\Delta\xi_2\, h_3(\xi_1 + \Delta\xi_1, \overline{\xi_2}, \overline{\xi_3})\Delta\xi_3 -$$
$$A_1(\xi_1, \overline{\xi_2}, \overline{\xi_3})\, h_2(\xi_1, \overline{\xi_2}, \overline{\xi_3})\Delta\xi_2\, h_3(\xi_1, \overline{\xi_2}, \overline{\xi_3})\Delta\xi_3\big] / (h_1\Delta\xi_1 h_2\Delta\xi_2 h_3\Delta\xi_3)$$
$$\to \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_1}(h_2h_3A_1)$$

$\Rightarrow$ we get for the divergence

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\xi_1}(h_2h_3A_1) + \frac{\partial}{\partial\xi_2}(h_3h_1A_2) + \frac{\partial}{\partial\xi_3}(h_1h_2A_3)\right\}}$$

The Laplacian operator follows ($\boldsymbol{\nabla}^2 = \nabla^2 = \Delta$):

$$\boxed{\Delta = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\xi_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial\xi_3}\right)\right\}}$$

Curl

Using a similar derivation as in Cartesian coordinates we get for the curl of vector $\mathbf{A}$

$$\nabla \times \mathbf{A} = \frac{1}{h_2 h_3} \left\{ \frac{\partial}{\partial \xi_2}(h_3 A_3) - \frac{\partial}{\partial \xi_3}(h_2 A_2) \right\} \mathbf{e}_{\xi_1} + \text{ cyclic permutations}$$

Representation as determinant:

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\,\mathbf{e}_{\xi_1} & h_2\,\mathbf{e}_{\xi_2} & h_3\,\mathbf{e}_{\xi_3} \\ \dfrac{\partial}{\partial \xi_1} & \dfrac{\partial}{\partial \xi_2} & \dfrac{\partial}{\partial \xi_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

A.4 Some explicit forms of vector operations

Cartesian coordinates (x_1, x_2, x_3) **with base vectors** $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$
\begin{aligned}
\boldsymbol{\nabla}\psi &= \frac{\partial\psi}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial\psi}{\partial x_2}\,\mathbf{e}_2 + \frac{\partial\psi}{\partial x_3}\,\mathbf{e}_3 \\
\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\
\boldsymbol{\nabla}\times\mathbf{A} &= \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\mathbf{e}_3 \\
\Delta\,\psi &= \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}
\end{aligned}
$$

Cylindrical coordinates (ρ, φ, z) **with base vectors** $\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{e}_z$

$$
\begin{aligned}
\boldsymbol{\nabla}\psi &= \frac{\partial\psi}{\partial\rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\varphi}\,\mathbf{e}_\varphi + \frac{\partial\psi}{\partial z}\mathbf{e}_z \\
\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\,(\rho\,A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \\
\boldsymbol{\nabla}\times\mathbf{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\mathbf{e}_\varphi + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho\,A_\varphi) - \frac{\partial A_\rho}{\partial\varphi}\right)\mathbf{e}_z \\
\Delta\,\psi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}
\end{aligned}
$$

Spherical coordinates (r, θ, φ) **with base vectors** $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$

$$
\begin{aligned}
\boldsymbol{\nabla}\psi &= \frac{\partial\psi}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\mathbf{e}_\theta + \frac{1}{r\,\sin\theta}\,\frac{\partial\psi}{\partial\varphi}\mathbf{e}_\varphi \\
\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\,(r^2 A_r) + \frac{1}{r\,\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\,A_\theta) + \frac{1}{r\,\sin\theta}\,\frac{\partial A_\varphi}{\partial\varphi} \\
\boldsymbol{\nabla}\times\mathbf{A} &= \frac{1}{r\,\sin\theta}\left(\frac{\partial}{\partial\theta}\,(\sin\theta\,A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right)\mathbf{e}_r \\
&+ \left(\frac{1}{r\,\sin\theta}\,\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\,(r\,A_\varphi)\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\,(r\,A_\theta) - \frac{\partial A_r}{\partial\theta}\right)\mathbf{e}_\varphi \\
\Delta\,\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\,\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\,\sin^2\theta}\,\frac{\partial^2\psi}{\partial\varphi^2}
\end{aligned}
$$

Note

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}\,(r\,\psi)
$$

A.5 Relation between unit base vectors and their time derivatives

Relation between the local unit base vectors of cylindrical coordinates $\mathbf{e}_\rho(t), \mathbf{e}_\varphi(t), \mathbf{e}_z$ and their time derivatives and the global constant unit base vectors of Cartesian coordinates $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\begin{aligned} \mathbf{e}_\rho &= \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2 \\ \mathbf{e}_\varphi &= -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2 \\ \mathbf{e}_z &= \mathbf{e}_3 \end{aligned} \qquad \begin{aligned} \mathbf{e}_1 &= \cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi \\ \mathbf{e}_2 &= \sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi \\ \mathbf{e}_3 &= \mathbf{e}_z \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{e}}_\rho &= -\sin\varphi\,\dot\varphi\,\mathbf{e}_1 + \cos\varphi\,\dot\varphi\,\mathbf{e}_2 = \dot\varphi\,\mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\varphi &= -\dot\varphi\,\mathbf{e}_\rho \\ \dot{\mathbf{e}}_z &= 0 \end{aligned} \qquad \begin{aligned} \dot{\mathbf{e}}_1 &= 0 \\ \dot{\mathbf{e}}_2 &= 0 \\ \dot{\mathbf{e}}_3 &= 0 \end{aligned}$$

Same for the local unit base vectors of spherical coordinates $\mathbf{e}_r(t), \mathbf{e}_\theta(t), \mathbf{e}_\varphi(t)$ and their time derivatives

$$\begin{aligned} \mathbf{e}_r &= \sin\theta\,\cos\varphi\,\mathbf{e}_1 + \sin\theta\,\sin\varphi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3 \\ \mathbf{e}_\theta &= \cos\theta\,\cos\varphi\,\mathbf{e}_1 + \cos\theta\,\sin\varphi\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_3 \\ \mathbf{e}_\varphi &= -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2 \end{aligned}$$

$$\begin{aligned} \mathbf{e}_1 &= \sin\theta\,\cos\varphi\,\mathbf{e}_r + \cos\theta\,\cos\varphi\,\mathbf{e}_\theta - \sin\varphi\,\mathbf{e}_\varphi \\ \mathbf{e}_2 &= \sin\theta\,\sin\varphi\,\mathbf{e}_r + \cos\theta\,\sin\varphi\,\mathbf{e}_\theta + \cos\varphi\,\mathbf{e}_\varphi \\ \mathbf{e}_3 &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{e}}_r &= \dot\theta\,\mathbf{e}_\theta + \sin\theta\,\dot\varphi\,\mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\theta &= -\dot\theta\,\mathbf{e}_r + \cos\theta\,\dot\varphi\,\mathbf{e}_\varphi \\ \dot{\mathbf{e}}_\varphi &= -\dot\varphi\,(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \end{aligned}$$

Relation between the local unit base vectors $\mathbf{e}_\rho(t), \mathbf{e}_\varphi(t), \mathbf{e}_z$ and $\mathbf{e}_r(t), \mathbf{e}_\theta(t), \mathbf{e}_\varphi(t)$

$$\begin{aligned} \mathbf{e}_\rho &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \\ \mathbf{e}_\varphi &= \mathbf{e}_\varphi \\ \mathbf{e}_z &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \end{aligned} \qquad \begin{aligned} \mathbf{e}_r &= \sin\theta\,\mathbf{e}_\rho + \cos\theta\,\mathbf{e}_z \\ \mathbf{e}_\theta &= \cos\theta\,\mathbf{e}_\rho - \sin\theta\,\mathbf{e}_z \\ \mathbf{e}_\varphi &= \mathbf{e}_\varphi \end{aligned}$$