Chapter 1

Some elements of vector and tensor analysis and the Dirac δ-function

The vector analysis is useful in physics
formulate the laws of physics independently of any preferred direction in space
experimentally known: the laws of mechanics are independent of choosing a right-handed or left-handed system of coordinate axes

1.1 Orthogonal transformations and tensors

1.1.1 The radius-vector and orthogonal transformations

In ordinary 3-dimensional space – called Euclidean space – let us introduce a coordinate system K with some origin O, Cartesian coordinates $(x_1, x_2, x_3) = (x, y, z)$ and base unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ along the Cartesian directions
The $\mathbf{e}_i$ are orthogonal to each other, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a right-handed system
the coordinates x_i are the components of the <u>radius-vector</u> or coordinate-vector of a point particle

$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3 \equiv (x_1, x_2, x_3)$$

the magnitude of the radius-vector is

$$|\mathbf{x}| \equiv x = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Consider a rotated around the origin O coordinate system K' with (x'_1, x'_2, x'_3) with unit base vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$
The radius vector $\mathbf{x}$ is defined independently from the chosen coordinate system

$$\Rightarrow \quad \mathbf{x} = \sum_{i=1}^{3} x_i\,\mathbf{e}_i = \sum_{i=1}^{3} x'_i\,\mathbf{e}'_i = \mathbf{x}'$$

Since under

$$\mathbf{x} \to \lambda\,\mathbf{x} \quad \Rightarrow \quad x_i \to \lambda\, x_i\,, x'_i \to \lambda\, x'_i$$

the relation between x'_i and x_j must be linear

$$x'_i = \sum_{j=1}^{3} a_{ij}\, x_j\,, \quad a_{ij} \text{ real coefficients}$$

and

$$x^2 = |\mathbf{x}|^2 = \mathbf{x}^2 = \mathbf{x}'^2 \quad \Rightarrow \quad \sum_{i=1}^{3} x_i^2 = \sum_{j=1}^{3} x'^2_j$$

this transformation is called orthogonal

1.1.2 Classification of physical quantities as rotational tensors

We consider nonrelativistic (!) physical quantities

Use Einstein's summation convention
summation over double appearing (dummy) indices is understood
Latin indices $i, j, k, \dots = 1, 2, 3$

A linear transformation of coordinates of a point such that the sum of squares of coordinates remains invariant is called an orthogonal transformations of coordinates

$$x'_i = a_{ij} x_j\,, \quad \mathbf{x}'^2 = \mathbf{x}^2\,, \quad a_{ij} \text{ real coefficients} \tag{1.1}$$

From

$$x'^2_i = a_{ij}\, x_j\, a_{ik}\, x_k = x_j^2 \quad \Rightarrow \quad a_{ij} a_{ik} = \delta_{jk}$$

Kronecker's symbol

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

The 3×3 numbers a_{ik} can be represented as matrix a, the 3 numbers x_i as column vector $\mathbf{x}$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Denote the transposition of a matrix a by $\widetilde{a}$ (a transposed column vector becomes a raw vector)
In this matrix notation the orthogonality takes the form

$$\widetilde{a}a = a\widetilde{a} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \widetilde{a} = a^{-1}$$

I denotes the unit matrix
The raw vectors and column vectors of the orthogonal matrix a are orthonormalised
Example: rotation about the $x_3 = z$ axis in anticlockwise manner by angle φ

$$a = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Backtransformation

$$x' = ax \quad \Rightarrow \quad \widetilde{a}x' = \widetilde{a}ax = x \text{ or } \widetilde{x} = \widetilde{x'}a$$

in index notation with $(a^{-1})_{ij} = (\widetilde{a})_{ij} = a_{ji}$

$$x_i = (\widetilde{a})_{ij}\, x'_j = a_{ji}\, x'_j$$

Taking the trace

$$\det(\widetilde{a}a) = [\det(a)]^2 = 1$$

$\det(a) = +1$ — proper rotation
$\det(a) = -1$ — improper rotation: reflection + rotation

Nonrelativistic physical quantities are classified as rotational tensor of various ranks depending on how they transform under rotation

$\mathbf{x}_\alpha$, $\mathbf{v}_\alpha$, $\mathbf{p}_\alpha$ for particle α transform according to (1.1) as coordinates: **vectors**
$\mathbf{x}_1 \cdot \mathbf{x}_2$, $\mathbf{v}_1 \cdot \mathbf{p}_2$ invariant under rotations: **scalars**
groups of nine quantities that transform according to

$$B'_{ij} = a_{ik}a_{jl}\, B_{kl} \qquad \textbf{second rank tensors, tensors}$$

Tensor of rank N

$$T'_{i_1\cdots i_N} = a_{i_1 j_1} \cdots a_{i_N j_N}\, T_{j_1\cdots j_N}$$

Construction of new tensors

- Addition (a, b numbers)

$$a\, S_{i_1\cdots i_N} + b\, T_{j_1\cdots j_N} \qquad \text{tensor of rank } N$$

- Multiplication

$$S_{i_1\cdots i_N}\, T_{j_1\cdots j_N} \qquad \text{tensor of rank } N + M$$

- Contraction

$$S_{i_1\cdots j \cdots j \cdots i_N} \qquad \text{tensor of rank } N - 2$$

Note $V_i W_i$ and $\mathbf{r}^2 = x_i^2$ form scalars

Tensors are defined by their behavior under orthogonal transformations (we used Cartesian coordinates in 3-dim Euclidean space)
⇒ Equations for tensor quantities are *forminvariant* (called *covariant*) under orthogonal transformations
Therefore, write physical equations in form of covariant equations of tensors of various ranks

1.1.3 Further notations

Kronecker's symbol

The Kronecker's symbol is defined to be independent from the chosen Cartesian coordinate system

$$\delta'_{ik} = \delta_{ik}$$

it fulfills the definition of a tensor of rank 2

$$\delta'_{ik} = a_{in} a_{km} \delta_{nm} = a_{in} a_{kn} = \delta_{ik}$$

matrix $I = (\delta_{ik})$ is the unit matrix
Useful:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}\,, \quad A_i = \delta_{ij}\, A_j\,, \quad \mathbf{A}^2 = \delta_{ij}\, A_i A_j$$

Totally antisymmetric tensor of rank 3

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any of indices } i, j, k \text{ equal} \\ 1, & \text{if } i \neq j \neq k \text{ reached in even permutations from 123} \\ -1, & \text{if } i \neq j \neq k \text{ reached in odd permutations from 123} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1\,, \quad \varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$$

$$\varepsilon_{ijk}\, \varepsilon_{i'j'k'} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji'} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki'} & \delta_{kj'} & \delta_{kk'} \end{vmatrix}$$

$$\varepsilon_{ijk}\, \varepsilon_{ilm} = \delta_{jl}\, \delta_{km} - \delta_{jm}\, \delta_{kl}\,, \quad \varepsilon_{ijk}\, \varepsilon_{ijl} = 2\delta_{jl}\,, \quad \varepsilon_{ijk}\, \varepsilon_{ijk} = 3!$$

Vector formulae

$$
\begin{aligned}
(\mathbf{A}\times\mathbf{B})_i &= \varepsilon_{ijk}\, A_j B_k \\
(\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))_i &= \varepsilon_{ijk}\, A_j\, (\mathbf{B}\times\mathbf{C})_k = \varepsilon_{ijk}\, A_j\, \varepsilon_{klm}\, B_l\, C_m = \varepsilon_{kij}\, \varepsilon_{klm}\, A_j\, B_l\, C_m \\
&= (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, A_j\, B_l\, C_m = A_m\, B_i\, C_m - A_l\, B_l\, C_i \\
&= (\mathbf{A}\cdot\mathbf{C})\, B_i - (\mathbf{A}\cdot\mathbf{B})\, C_i
\end{aligned}
$$

$$
\begin{gathered}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \\
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C} \\
(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})\,(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})\,(\mathbf{B}\cdot\mathbf{C})
\end{gathered}
$$

Vector differentiation

Consider a vector $\mathbf{A}(x)$ depending on the scalar argument x

$$\mathbf{A}(x) = A(x)\,\mathbf{e}_A\,, \quad A(x) = |\mathbf{A}(x)|\,, \quad \mathbf{e}_A = \frac{\mathbf{A}(x)}{A(x)}$$

derivative with respect to the argument x

$$\frac{d\mathbf{A}(x)}{dx} = \frac{d}{dx}\big[A(x)\,\mathbf{e}_A(x)\big] = \frac{dA}{dx}\,\mathbf{e}_A + A\,\frac{d\mathbf{e}_A}{dx} = \frac{dA}{dx}\,\mathbf{e}_A + A\,\left|\frac{d\mathbf{e}_A}{dx}\right|\,\mathbf{e}_B\,, \quad \mathbf{e}_B \perp \mathbf{e}_A$$

the change of the unit vector $\dfrac{d\mathbf{e}_A}{dx} \perp \mathbf{e}_A$

Total time derivative of a vector $\mathbf{A}(x_1, x_2, x_3, t)$ depending on Cartesian coordinates x_i and time t

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial x_1}\,\frac{dx_1}{dt} + \frac{\partial\mathbf{A}}{\partial x_2}\,\frac{dx_2}{dt} + \frac{\partial\mathbf{A}}{\partial x_3}\,\frac{dx_3}{dt} + \frac{\partial\mathbf{A}}{\partial t} = \frac{\partial\mathbf{A}}{\partial x_1}\,\dot{x}_1 + \frac{\partial\mathbf{A}}{\partial x_2}\,\dot{x}_2 + \frac{\partial\mathbf{A}}{\partial x_3}\,\dot{x}_3 + \frac{\partial\mathbf{A}}{\partial t}$$

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \dot{x}_i\,\frac{\partial}{\partial x_i}\,\mathbf{A}$$

This can be written in vector form (covariant form) using the nabla operator $\boldsymbol{\nabla}$ defined below in Chapter 1.2.4

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + (\dot{\mathbf{r}}\cdot\boldsymbol{\nabla})\,\mathbf{A} \tag{1.2}$$

Tensor fields under rotations

We use an “active” view of rotation:
Coordinate axes are fixed, the physical system undergoes a rotation $\mathbf{x}_{(\alpha)} \to \mathbf{x}'_{(\alpha)}$

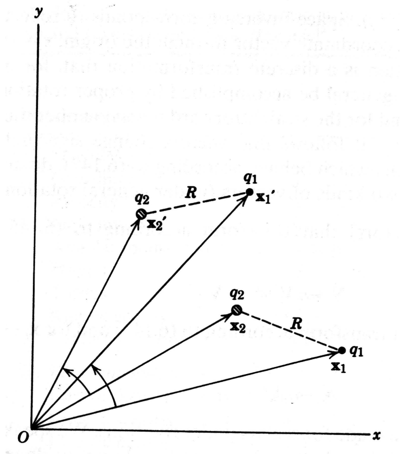

Figure 1.1: Active rotation of a system of two point charges

Under rotation of tensor fields (scalar,vector, ...fields) also transformation of the position components
scalar function $\Phi(\mathbf{x}_\alpha)$ under rotation

$$\Phi'(\mathbf{x}'_\alpha) = \Phi(\mathbf{x}_\alpha)$$

components of a vector function $V_i(\mathbf{x}_\alpha)$ under rotation

$$V'_i(\mathbf{x}'_\alpha) = a_{ij}\, V_j(\mathbf{x}_\alpha)$$

analogously higher rank tensor functions under rotation

$$T'_{i'j'\ldots m'}(\mathbf{x}'_\alpha) = a_{i'i}\, a_{j'j} \ldots a_{m'm}\, T_{ij\ldots m}(\mathbf{x}_\alpha)$$

The cross product of vectors/vector fields

$$\mathbf{A} = \mathbf{B} \times \mathbf{C}$$

presence of two vectors on the right hand side
cross product has some attributes of a traceless antisymmetric second-rank tensor (with 3 independent components)
treated as vector with respect to $x'_i = a_{ij}x_j$
the real transformation for the cross product is

$$A'_i = \det(a)\, a_{ij} A_j$$

Under proper rotations $\det(a) = 1$, the cross product transforms as a vector

Spatial reflections or inversions

- Spatial reflection in plane
  change signs of the normal components of coordinate-vectors of all particles α relative to a plane
  leave components $\|$ to plane unchanged
  reflection in the (x_1, x_2) plane:

$$\mathbf{x}_\alpha = (x_{\alpha\,1}, x_{\alpha\,2}, x_{\alpha\,3}) \to \mathbf{x}'_\alpha = (x_{\alpha\,1}, x_{\alpha\,2}, -x_{\alpha\,3})$$

- Space inversion
  reflection of all three components of every coordinate vector through the origin

$$\mathbf{x}_\alpha = (x_{\alpha\,1}, x_{\alpha\,2}, x_{\alpha\,3}) \to \mathbf{x}'_\alpha = (-x_{\alpha\,1}, -x_{\alpha\,2}, -x_{\alpha\,3})$$

spatial reflection is a <u>discrete</u> transformation
corresponds to $\det(a) = -1$

$$x'_{\alpha\,i} = a_{ij}\, x_{\alpha\,j} \quad \text{with } a_{ij} = -\delta_{ij}$$

vectors change sign under spatial inversion, cross products do not change sign
$\Rightarrow$ distinguish two kind of vectors under spatial inversions:
For $\mathbf{x} = (x_1, x_2, x_3) \to \mathbf{x}' = -\mathbf{x} = (-x_1, -x_2, -x_3)$

$$\begin{array}{lcll} \mathbf{V} & \to & \mathbf{V}' = -\mathbf{V} & \underline{\text{polar vectors}} \\ \mathbf{A} & \to & \mathbf{A}' = \mathbf{A} & \underline{\text{axial vectors or pseudovectors}} \end{array}$$

Similar classification for scalars under rotation and spatial inversion:
<u>scalars</u> – do not change sign
<u>pseudoscalars</u> – do change sign (Ex.: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ polar vectors)

Tensor of rank N
transformation property under inversion can be deduced directly if the tensor is built up as products of components of polar or axial vectors
if tensor transforms with factor $(-1)^N$ – true tensor or <u>tensor of rank N</u>
if tensor transforms with factor $(-1)^{N+1}$ – <u>pseudotensor of rank N</u>

Note the special tensors:
δ_{ik} is a true tensor of rank 2
ε_{ijk} is a pseudotensor of rank 3 (ε_{ijk} enters in the definition of the vector product: the vector product of two polar vectors leads to an axial vector)

1.2 Vector differential operators

Consider a region of 3-dimensional Euclidean space R with a scalar field $\psi(\mathbf{x})$ and a vector field $\mathbf{A}(\mathbf{x})$ (suppress the t-dependence)
the region R is regular, $\psi(\mathbf{x}), \mathbf{A}(\mathbf{x})$ and their derivatives are continuous
$\rightarrow \psi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ are continuously differentiable in R
3 differential operations

1.2.1 Gradient of a scalar field

The gradient of a scalar field $\psi(\mathbf{x})$, denoted by $\operatorname{grad}\psi(\mathbf{x})$, is a vector field at position $\mathbf{x}$
the component of the vector field $\operatorname{grad}\psi$ in direction of an arbitrary unit vector $\mathbf{n}$ is defined as ($\Delta\mathbf{x} = \mathbf{n}\Delta x$)

$$\mathbf{n}\cdot\operatorname{grad}\psi(\mathbf{x}) = \lim_{\Delta x\to 0}\frac{\psi(\mathbf{r}+\mathbf{n}\Delta x)-\psi(\mathbf{x})}{\Delta x}$$

$\mathbf{n}\cdot\operatorname{grad}\psi$ is the derivative of ψ in the direction $\mathbf{n}$

$$\mathbf{n}\cdot\operatorname{grad}\psi = \frac{\partial\psi}{\partial n}$$

Geometric interpretation:

the equation $\psi(\mathbf{x}) = \text{const}$ represents a surface
the vector $\operatorname{grad}\psi$ at position $\mathbf{x}$ is $\perp$ to that surface and points in the direction of the greatest rate of increase of ψ
$|\operatorname{grad}\psi|$ is the greatest rate of change
For Cartesian coordinates:
for the choice $\mathbf{n} = \mathbf{e}_1$, $\mathbf{n}\Delta x \rightarrow \mathbf{e}_1\Delta x_1$, $\Delta x \rightarrow \Delta x_1$ with x_2, x_3 fixed

$$\mathbf{e}_1\cdot\operatorname{grad}\Phi(x_1,x_2,x_3) = \lim_{\Delta x_1\to 0}\frac{\psi(x_1+\Delta x_1,x_2,x_3)-\psi(x_1,x_2,x_3)}{\Delta x_1} \rightarrow \frac{\partial\psi}{\partial x_1}$$

$$\Rightarrow \quad \operatorname{grad}\psi \quad = \quad \frac{\partial\psi}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial\psi}{\partial x_2}\,\mathbf{e}_2 + \frac{\partial\psi}{\partial x_3}\,\mathbf{e}_3$$

1.2.2 Divergence of a vector field

The divergence of a vector field is a scalar field that measures the magnitude of a vector's field source or sink at a given point $\mathbf{x}$
definition

$$\operatorname{div}\mathbf{A}(\mathbf{x}) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \mathbf{A} \cdot \mathbf{n}\, da$$

volume ΔV is bounded by the closed surface S, with area element da and unit outward normal $\mathbf{n}$ at da
in the limiting procedure $\Delta V \to 0$ the point $\mathbf{x}$ is always contained inside S

If $\operatorname{div}\mathbf{A} > (<)0$, the vector field has a source (sink)
$\operatorname{div}\mathbf{A}$ measures the strength of the source/sink
Illustration: for a sphere

In Cartesian coordinates:
consider a rectangular box with volume $\Delta V = \Delta x_1\, \Delta x_2\, \Delta x_3$ starting from point (x_1, x_2, x_3) and sum over the six surface areas in the limit of vanishing volume [the convenient positions in integration region $\overline{x_i} \to x_i$ according to the mean value theorem of integration]

$$\operatorname{div}\mathbf{A}(x_1, x_2, x_3) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \Big\{ A_1\, dx_2\, dx_3 + A_2\, dx_3\, dx_1 + A_3\, dx_1\, dx_2 \Big\} =$$

$$\lim_{\Delta V \to 0} \frac{\Big\{ \Delta x_2\, \Delta x_3\, [A_1(x_1 + \Delta x_1, \overline{x_2}, \overline{x_3}) - A_1(x_1, \overline{x_2}, \overline{x_3}] + \ldots \Big\}}{\Delta x_1\, \Delta x_2\, \Delta x_3} \to \frac{\partial A_1}{\partial x_1} + \ldots$$

$$\Rightarrow \quad \operatorname{div}\mathbf{A} \quad = \quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_2}{\partial x_3}$$

1.2.3 Curl or rotation of a vector field

The curl is a vector field, denoted by $\mathrm{curl}\,\mathbf{A}(\mathbf{x}) \equiv \mathrm{rot}\,\mathbf{A}(\mathbf{x})$ describing the infinitesimal rotation of a 3-dimensional vector field ay position $\mathbf{x}$ (in fluid dynamics called a vortex) the component of $\mathrm{curl}\,\mathbf{A}$ in direction of an arbitrary unit vector $\mathbf{n}$ is defined as

$$\mathrm{curl}\,\mathbf{A}(\mathbf{x}) \cdot \mathbf{n} \equiv \mathrm{rot}\,\mathbf{A}(\mathbf{x}) \cdot \mathbf{n} = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \oint_C \mathbf{A} \cdot d\mathbf{l}$$

The direction of the open oriented surface in the limit $\Delta a \to 0$ is parallel to $\mathbf{n}$. The contour C bends Δa with the normal $\mathbf{n}$ defined by the right-hand-screw rule in relation to the sense of the line integral around C
$\oint_C \mathbf{A} \cdot d\mathbf{l}$ is the line integral along the boundary of the area in question
In the limit $\Delta a \to 0$ the direction of $\mathrm{curl}\,\mathbf{A}(\mathbf{r})$ is given by $\mathbf{n}$

For illustration choose circle with normal $\perp$ to plane and the vector field $\mathbf{A}$ in the plane

In Cartesian coordinates:
choose $\mathbf{n} = \mathbf{e}_3$ and rectangle in the (x_1, x_2) plane starting from point (x_1, x_2, x_3) with area $\Delta a = \Delta x_1\, \Delta x_2$
sum over all sides of the line integral and divide by that area
we get for the x_3 component of the curl

$$(\mathrm{curl}\,\mathbf{A}(x_1, x_2, x_3))_3 = \lim_{\Delta a \to 0}$$
$$\frac{\Delta x_1\, [A_1(\overline{x_1}, x_2, x_3) - A_1(\overline{x_1}, x_2 + \Delta x_2, x_3)] + \Delta x_2\, [A_2(x_1 + \Delta x_1, \overline{x_2}, x_3) - A_2(x_1, \overline{x_2}, x_3)]}{\Delta x_1\, \Delta x_2}$$
$$\to \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$

As result we get

$$\Rightarrow \quad \mathrm{curl}\,\mathbf{A} \quad = \quad \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) \mathbf{e}_1 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) \mathbf{e}_2 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) \mathbf{e}_3$$

Another form in Cartesian coordinates in form of a determinant

$$\mathrm{curl}\,\mathbf{A} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

1.2.4 Nabla operator

All definitions were made independently of the choice of the coordinate system

in Cartesian coordinates we got

$$
\begin{aligned}
\operatorname{grad}\psi &= \frac{\partial\psi}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial\psi}{\partial x_2}\,\mathbf{e}_2 + \frac{\partial\psi}{\partial x_3}\,\mathbf{e}_3 \\
\operatorname{div}\mathbf{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_2}{\partial x_3} \\
\operatorname{curl}\mathbf{A} &= \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\mathbf{e}_3
\end{aligned}
$$

The partial derivatives suggest to write the form of the differential operators using the vector differential operator $\boldsymbol{\nabla}$

$$\boldsymbol{\nabla} = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}$$

$$\operatorname{grad}\psi \equiv \boldsymbol{\nabla}\psi\,, \quad \operatorname{div}\mathbf{A} \equiv \boldsymbol{\nabla}\cdot\mathbf{A}\,, \quad \operatorname{curl}\mathbf{A} \equiv \boldsymbol{\nabla}\times\mathbf{A}$$

The components of $\boldsymbol{\nabla}$ transform as a vector under orthogonal transformations

$$\frac{\partial}{\partial x_i} = a_{ij}\,\frac{\partial}{\partial x_j}$$

This also explains the use of $\boldsymbol{\nabla}$ in eq. (1.2)
We will use from now mainly the $\boldsymbol{\nabla}$-notation which is independent of the coordinate system for the exact definition of $\boldsymbol{\nabla}$ in other curvilinear orthogonal coordinates see Chapter A.1

Note the following relations

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi &= \operatorname{div}(\operatorname{grad}\psi) = \boldsymbol{\nabla}^2\psi = \nabla^2\psi = \Delta\psi \\
\boldsymbol{\nabla}\times\boldsymbol{\nabla}\psi &= \operatorname{curl}(\operatorname{grad}\psi) \equiv 0 \\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) &= \operatorname{div}(\operatorname{curl}\mathbf{A}) \equiv 0 \\
\Delta\mathbf{A} &= \operatorname{grad}(\operatorname{div}\mathbf{A}) - \operatorname{curl}(\operatorname{curl}\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})
\end{aligned}
$$

Index notations (in Cartesian coordinates, $\partial_i \equiv \partial/\partial x_i$)

$$(\boldsymbol{\nabla}\psi)_i = \partial_i\psi\,, \quad \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi = \partial_i^2\psi\,, \quad (\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \partial_i\partial_k A_k - \partial_k^2 A_i$$

if $\boldsymbol{\nabla}\times\mathbf{A} = 0$: $\mathbf{A}$ irrotational vector field
an irrotational vector is expressible as gradient of a scalar function

if $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$: $\mathbf{A}$ solenoidal vector field
an solenoidal vector is expressible as curl of a vector function

1.3 Integral theorems

The results of a partial integration of the differential operators are called integral theorems ψ, φ and $\mathbf{A}$ are well-behaved scalar or vector functions (fields), T_{ij} is a well behaved tensor field of second rank

1.3.1 Divergence Theorem (Gauss', Green's or Ostrogradsky's Theorem)

V: three-dimensional volume with volume element d^3x
S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da ($d\mathbf{a} = da\,\mathbf{n}$), follows directly from the definition of $\boldsymbol{\nabla} \cdot \mathbf{A}$

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{A}\, d^3x = \oint_S \mathbf{A} \cdot \mathbf{n}\, da$$

Variants of the divergence theorem (derivation of Green's identities, see below Chapter 2.5)

$$\begin{aligned}
\int_V \boldsymbol{\nabla}\psi\, d^3x &= \oint_S \psi\, \mathbf{n}\, da \\
\int_V \boldsymbol{\nabla} \times \mathbf{A}\, d^3x &= \oint_S \mathbf{n} \times \mathbf{A}\, da \\
\int_V \frac{\partial}{\partial x_j} T_{ij}\, d^3x &= \oint_S T_{ij} n_j\, da \quad \text{(in Cartesian coordinates)} \\
\int_V \left(\varphi \boldsymbol{\nabla}^2 \psi + \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\psi\right) d^3x &= \oint_S \varphi \frac{\partial \psi}{\partial n}\, da \quad \left(\text{Green's 1st identity, } \frac{\partial \psi}{\partial n} = \mathbf{n} \cdot \boldsymbol{\nabla}\psi\right) \\
\int_V \left(\varphi \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \varphi\right) d^3x &= \oint_S \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right) da \quad \text{(Green's 2nd identity or theorem)}
\end{aligned}$$

1.3.2 Stokes' Theorem

The Stokes' theorem follows directly from the definition $(\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n}$
S denotes an <u>open</u> surface, C contour bending S, $d\mathbf{l}$ vector of infinitesimal length element along C
normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C

$$\begin{aligned}
\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n}\, da &= \oint_C \mathbf{A} \cdot d\mathbf{l} \\
\int_S \mathbf{n} \times \boldsymbol{\nabla}\psi\, da &= \oint_C \psi\, d\mathbf{l} \qquad \text{(variant)}
\end{aligned}$$

The variant is obtained by multiplying the line integral with a constant vector and applying the standard Stokes' theorem (similar procedure for the first two variants of the divergence theorem)
Stokes' theorem implies: an irrotational vector field $\mathbf{A}$ with $\nabla \times \mathbf{A} = 0$ can be always represented via the gradient of a scalar field function $\mathbf{A} = -\boldsymbol{\nabla}\varphi$

1.4 The Dirac δ-function

1.4.1 Definitions

Consider a function $d_l(x)$ depending on a parameter l

$$d_l(x) = \begin{cases} \dfrac{1}{l} & -l/2 \le x \le l/2 \\ 0 & \text{otherwise} \end{cases}$$

for an arbitrary continuous test function $f(x)$ we calculate the convolution integral with $d_l(x)$ and after integration let go $l \to 0$ using the mean value theorem of integration

$$\lim_{l\to 0} \int_{-\infty}^{\infty} dx\, d_l(x - x_0)\, f(x) = \lim_{l\to 0} \frac{1}{l} \int_{x_0-l/2}^{x_0+l/2} dx\, f(x) = \lim_{l\to 0} f(\bar{x}) = f(x_0)$$

For the convolution we use the abbreviation

$$\lim_{l\to 0} \int_{-\infty}^{\infty} dx\, d_l(x - x_0)\, f(x) = \int_{-\infty}^{\infty} dx\, \delta(x - x_0)\, f(x)$$

with the δ-function (limit after integration!)

$$\delta(x) = \lim_{l\to 0} d_l(x)$$

One can construct other functions $d_l(x)$:
sufficient $d_l(x) > 0$, localized in a region l around $x = 0$ and normalized to 1

$$\int_{-\infty}^{\infty} dx\, d_l(x) = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Examples for $d_l(x)$:

$$\frac{1}{\sqrt{\pi}\, l} \exp\left(-\frac{x^2}{l^2}\right), \quad \frac{1}{\pi} \frac{l}{x^2 + l^2}, \quad \frac{\sin \frac{x}{l}}{\pi\, x}$$

$\delta(x)$ is not a function in the usual sense
$\delta(x)$ and similar objects are called distributions
The distribution $\delta(x)$ (Dirac δ-function) is defined in such a way that for arbitrary continuous and integrable test functions $f(x)$ the following equation holds

$$\int_{-\infty}^{\infty} dx\, \delta(x - x_0)\, f(x) = f(x_0)$$

The independence of integrals on a very small size l in a physical application (e.g. volume integral over a test charge) is a simplification indicated by using the δ-function

from the definition it follows that the usual rules of integration are valid
Consider the convolution of

$$\delta'(x) = \frac{d}{dx}\, \delta(x)$$

with a test function $f(x)$ using partial integration

$$\int_{-\infty}^{\infty} dx\, \delta'(x-x_0)\, f(x) = -\int_{-\infty}^{\infty} dx\, \delta(x-x_0)\, f'(x) = -f'(x_0)$$

$\Rightarrow$ definition of distribution $\delta'(x)$
Step function $\Theta(x)$: integral over the δ-function

$$\int_{-\infty}^{x} dx'\, \delta(x'-x_0) = \Theta(x-x_0) = \begin{cases} 0 & \text{for } x < x_0 \\ 1 & \text{for } x > x_0 \end{cases}$$

the step function is an ordinary function
$\delta(x)$ can be written as derivative of the step function

$$\delta(x) = \Theta'(x) = \frac{d}{dx}\,\Theta(x)$$

Integral representation of the δ-function
Use the Fourier transformation ($f(x)$ continuous and square integrable)

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, g(k)\, \mathrm{e}^{\mathrm{i}\,kx} \\ g(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, \mathrm{e}^{-\mathrm{i}\,kx} \end{aligned}$$

Plug in $g(k)$ into $f(x)$ and replace x by x_0

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx_0 f(x_0) \int_{-\infty}^{\infty} dk\, \mathrm{e}^{\mathrm{i}\,k(x-x_0)}$$

$$\Rightarrow \quad \delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \mathrm{e}^{\mathrm{i}\,k(x-x_0)}$$

The Fourier transformation is based on the set of sin and cosine functions forming a complete set of orthogonal functions
Similar representations of the δ-function we get for every such set of orthogonal functions (see completeness conditions in Chapter 3.3)

1.4.2 Properties

- $\delta(x-a) = 0\,, \quad x \neq a$
- $\delta(-x) = \delta(x)\,, \quad x\,\delta(x) = 0$
- $\int \delta(x-a)\, dx = 1$ if $x = a$ included in integration region
- $\int f(x)\delta(x-a)\, dx = f(a)$ for arbitrary function $f(x)$

- $\delta'(-x) = -\delta'(x)\,, \quad x\,\delta'(x) = -\delta(x)$
- $\int f(x)\delta'(x-a)\,dx = -f'(a)$
- $\delta(a\,x) = \frac{1}{a}\delta(x)\,, a>0\,, \quad \delta(x^2-a^2) = \frac{1}{2a}\left[\delta(x-a)+\delta(x+a)\right]$
- for $f(x)$ with only simple zeros at $x = x_i$ in the integration region: $f(x_i) = 0$

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx}(x_i)\right|}$$

- 3-dim (in Cartesian coordinates)

$$\delta(\mathbf{x}-\mathbf{x}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

  vanishes everywhere except at $\mathbf{x} = \mathbf{x}_0$

- $$\int_{\Delta V} \delta(\mathbf{x}-\mathbf{x}_0)\,d^3x = \begin{cases} 1 & \text{if } \Delta V \text{ contains } \mathbf{x} = \mathbf{x}_0 \\ 0 & \text{if } \Delta V \text{ does not contain } \mathbf{x} = \mathbf{x}_0 \end{cases}$$

- 3-dim δ-function in cylindrical and spherical coordinates

$$\begin{aligned} \mathbf{x} = (\rho, \varphi, z)\,, & \qquad \mathbf{x}_0 = (\rho_0, \varphi_0, z_0) \\ \delta(\mathbf{x}-\mathbf{x}_0) &= \frac{1}{\rho_0}\,\delta(\rho-\rho_0)\,\delta(\varphi-\varphi_0)\,\delta(z-z_0) \end{aligned}$$

$$\begin{aligned} \mathbf{x} = (r, \theta, \varphi)\,, & \qquad \mathbf{x}_0 = (r_0, \theta_0, \varphi_0) \\ \delta(\mathbf{x}-\mathbf{x}_0) &= \frac{1}{r_0^2\,\sin\theta_0}\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\varphi-\varphi_0) \\ &= \frac{1}{r_0^2}\,\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\,\delta(\varphi-\varphi_0) \end{aligned}$$

- dimension of δ-function = dimension of inverse "volume"

Use δ-function to describe
charge density of a discrete set of charged point particles at positions $\mathbf{x}_i$
charge density: charges per unit volume

$$\boxed{\rho(\mathbf{x}) = \sum_{i=1}^{n} q_i\,\delta(\mathbf{x}-\mathbf{x}_i)}$$