One-loop renormalisation of quarks currents and first moments of DIS with overlap fermions and improved gauge action

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- Improved gluon propagator
- Overlap fermions, Feynman rules
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- Some results for renormalisation factors
- Summary

Introduction

- Relate lattice measurements of operators to physical quantities in continuum by renormalisation
- Lot of effort for better lattice formulations
- Lattice regularisation of chiral fermions available
  In particular, if Dirac operator satisfies Ginsparg-Wilson relation
  $$\gamma_5 D + D\gamma_5 = a\frac{1}{\rho}D\gamma_5 D$$
  lattice fermion action possesses an exact chiral symmetry
  Operator free of doublers, local in a more general sense, at least for sufficiently smooth gauge fields on the UV cutoff
  Use realisation by Neuberger operator D_N
- A RG improved gauge action expected to realize smooth gauge field fluctuations approximating those in the continuum limit better than with the plaquette action
- How to reduce systematic errors due to finite a ?
  Use actions with very small $\mathcal{O}(a^n)$ effects and choose coefficients according to RG arguments
  Usually with respect to this improvement quark and gluon parts of action treated differently
- Pure gauge Wilson action already improved to $\mathcal{O}(a)$
  Increase improvement to $\mathcal{O}(a^2)$ by adding loops with 6 links
  $\rightarrow$ improved gluon propagator needed in perturbation theory
- Quark sector with Wilson operator D_W:
  $\mathcal{O}(a)$ effects and lost chiral invariance
  $\rightarrow$ complicated mixing and mass acquisition under renormalisation for zero bare mass fermions
- Possible way out: use overlap fermions:
  solve chirality problem, action already improved to $\mathcal{O}(a)$

- In general, also improve measured operators

$$\bar{\psi}\gamma_{\{\mu}D_{\mu_1}\cdots D_{\mu_n\}}\psi$$

to same accuracy (they have $\mathcal{O}(a)$ corrections)

Improvement achieved for overlap fermions by rotating quark fields with operator $(1-\frac{1}{2}aD_N)$ similar to rotations using clover improvement

$$\bar{\psi}\left(1-\frac{1}{2}aD_N\right)\gamma_{\{\mu}D_{\mu_1}\cdots D_{\mu_n\}}\left(1-\frac{1}{2}aD_N\right)\psi$$

Fortunately: 1-loop renormalisation factors for improved and unimproved operators using massless quarks are the same

Reason: additive mass renormalisation is forbidden by chiral symmetry (Capitani et al. 1999):

$\rightarrow$ saves a lot in perturbative calculations

- In principle non-perturbative methods preferable to determine renormalisation factors

Why still use perturbation theory?

important as hint and guide for non–perturbative renormalisation studies

mixings (more intricate than in continuum) more transparent, especially if some amounts of mixings are small

window for non-perturbative signal difficult to obtain for some operators

- Some perturbative Z factors:

Overlap fermions and Wilson gauge action:
Quark bilinears (Alexandrou et al. (2000))
First moments of DIS structure functions (Capitani (2001))

Domain wall fermions and RG improved actions:
Quark bilinears (Aoki et al. (heplat 0206013))

- Here:
combine overlap fermions with RG improved actions
find perturbative Z factors of quark bilinears and first moments

Improved gluon propagator

General form of RG-improved gauge action including 6 links

$$S_{\text{gluon}} = \frac{1}{g^2}\Big\{c_0 \sum_{\text{plaquette}} \text{Tr}U_{pl} + c_1 \sum_{\text{rectangle}} \text{Tr}U_{rtg} + c_2 \sum_{\text{chair}} \text{Tr}U_{chr}$$

$$+ c_3 \sum_{\text{parallelogram}} \text{Tr}U_{plg}\Big\}$$

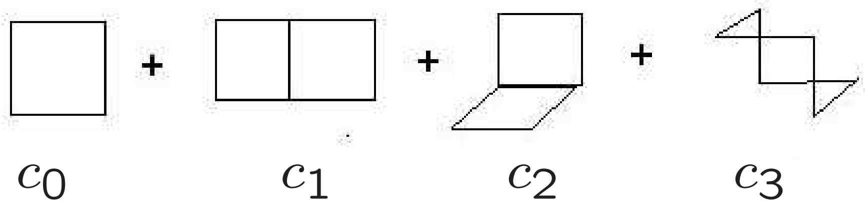

Figure 1: Gauge action contributions including six links

$c_0, \cdots, c_3$ satisfy $c_0 + 8c_1 + 16c_2 + 8c_3 = 1$

set parameters to value suggested by approximate RG analysis

Some discussed choices:

- $c_1 = -0.331$, $c_2 = c_3 = 0$ (Iwasaki)
- $c_1 = -0.27$, $c_2 + c_3 = -0.04$ (Iwasaki')
- $c_1 = -0.252$, $c_2 + c_3 = -0.17$ (Wilson)
- $c_1 = -1.40686$, $c_2 = c_3 = 0$ (DBW2: Takaishi, de Forcrand)
- $c_1 = -1/12$, $c_2 = c_3 = 0$ (Symanzik, Weisz, Lüscher)

Weak coupling perturbation theory:

write link variable in terms of gauge potential

$$U_{x,\mu} = \exp\left[igA_\mu(x + \frac{1}{2}\hat{\mu})\right]$$

expand in g

Use covariant gauge fixing with gauge parameter $\xi = 1 - \alpha$

Gluon propagator from free part of the gauge action

$$S_0^{\text{gluon}} = \frac{1}{2}\int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} A_\mu^a(k) \left(G_{\mu\nu}(k) - \frac{\xi}{\xi-1}\widehat{k}_\mu \widehat{k}_\nu \right) A_\nu^a(-k)$$

$$\begin{aligned} G_{\mu\nu}(k) &= \widehat{k}_\mu \widehat{k}_\nu + \sum_\rho \Big\{ \left(\widehat{k}_\rho^2 \delta_{\mu\nu} - \widehat{k}_\mu \widehat{k}_\rho \delta_{\rho\nu}\right)(1-\delta_{\mu\rho}) \times \\ &\qquad \left[1 - a^2 \left(C_1 \widehat{k}^2 + C_2(\widehat{k}_\mu^2 + \widehat{k}_\rho^2)\right)\right] \Big\} \end{aligned}$$

$$\widehat{k}_\mu = \frac{2}{a}\sin\frac{ak_\mu}{2}, \quad \widehat{k}^2 = \sum_\mu \widehat{k}_\mu^2$$

Coefficients C_i related to c_i's of improved action

$$C_1 = c_2 + c_3 , \quad C_2 = c_1 - c_2 - c_3$$

Solve orthogonality relation to derive gluon propagator

$$\sum_\rho \left(G_{\mu\rho} - \frac{\xi}{\xi-1}\widehat{k}_\mu \widehat{k}_\rho \right) \times D_{\rho\nu}(k) = \delta_{\mu\nu}$$

Exact solution only for $D = 2, 3, 4$ (Iwasaki, Weisz)

Need $D_{\mu\nu}$ in D-dimensional Euclidean space

$\rightarrow$ solve approximately orthogonality relation to accuracy of action

Find possible representation ($K_4 = \sum_\mu \widehat{k}_\mu^4$, neglect $\mathcal{O}(a^4)$)

$$D_{\mu\nu}^{\text{imp}} = \frac{1}{\widehat{k}^2}\left(\delta_{\mu\nu} - \xi\frac{\widehat{k}_\mu \widehat{k}_\nu}{\widehat{k}^2}\right) + \frac{a^2}{(\widehat{k}^2)^2} \times$$

$$\left\{ -\widehat{k}_\mu \widehat{k}_\nu \left[C_1 \widehat{k}^2 + C_2(\widehat{k}_\mu^2 + \widehat{k}_\nu^2)\right] + \delta_{\mu\nu}\left[C_1(\widehat{k}^2)^2 + C_2(K_4 + \widehat{k}^2 \widehat{k}_\nu^2)\right] \right\}$$

For $c_2 = c_3 = 0$ ($C_1 = 0$ and $C_2 = c_1$) improved gluon propagator coincides with $D_{\mu\nu}$ including $\mathcal{O}(a^2)$ given implicitly by Weisz (1983)

Overlap fermions, Feynman rules

Overlap Dirac operator of Neuberger (1993)

$$D_N = \frac{1}{a}\rho\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right], \;\; X = D_W - \frac{1}{a}\rho$$

in terms of the usual Wilson-Dirac operator

$$D_W = \frac{1}{2}\left[\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - ar\nabla^\star_\mu\nabla_\mu\right], \;\; \nabla_\mu\psi(x) = \frac{1}{a}\left[U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\right]$$

For $0 < \rho < 2r$ right spectrum of massless fermions

Feynman rules for overlap fermions, see Kikukawa, Yamada (1999) and Ishibashi et al. (2000)

Quark propagator

$$S(k) = \frac{-i\sum_\mu \gamma_\mu \sin ak_\mu}{2\rho\left(\omega(k) + b(k)\right)} + \frac{a}{2\rho}$$

with

$$\omega(k) = \sqrt{\frac{1}{a^2}\sum_\mu \sin^2 ak_\mu + b(k)^2}, \;\; b(k) = \frac{1}{a}\left(2r\hat{k}^2 - \rho\right)$$

Vertices for 1-loop calculations entirely given in terms of vertices of Wilson action and of the lowest order of X

Example: Quark-quark-gluon vertex (20 terms !)

$$V_{1\mu}(p_2,p_1) = \frac{\rho}{a}\frac{1}{\omega(p_2)+\omega(p_1)}\left[W_{1\mu}(p_2,p_1) - \frac{X_0(p_2)W^\dagger_{1\mu}(p_2,p_1)X_0(p_1)}{\omega(p_2)\omega(p_1)}\right]$$

$$\begin{aligned} W_{1\mu}(p_2,p_1) &= -g\left(i\gamma_\mu \cos\frac{a(p_1+p_2)_\mu}{2} + r\sin\frac{a(p_1+p_2)_\mu}{2}\right) \\ X_0(p) &= \frac{i}{a}\sum_\mu \gamma_\mu \sin ap_\mu + b(p) \end{aligned}$$

Feynman rules for lattice operators needed for moments of structure functions, see Göckeler et al. (1996)

About computation and program code

Typical diagrams to be calculated

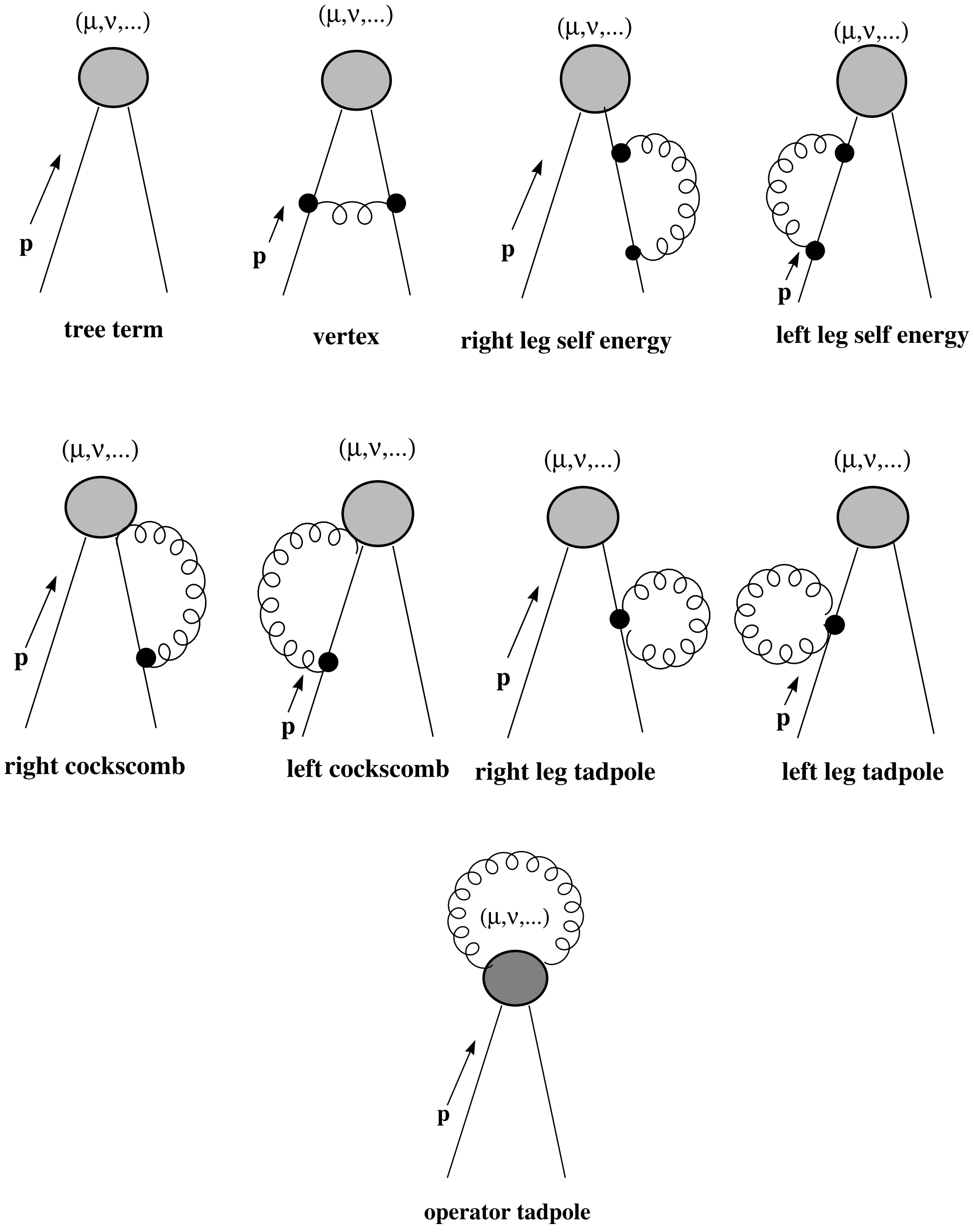

Figure 2: Lattice Feynman diagrams

Calculation: basically compute lattice loop integrals of the form

$$I_{\mu_1\cdots\mu_n}(a,p) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} K_{\mu_1\cdots\mu_n}(a,p,k)$$

K contains lattice quark and gluon propagators and sin, cos of lattice momenta p_μ and k_μ

Integration over first Brillouin zone $-\pi/a \leq k_\mu < \pi/a$

UV ($a \to 0, k \to \infty$) and IR ($k \to 0$) poles present

Calculation of I performed in two parts:
Decompose I (Kawai et al. 1981)

$$I = \widetilde{I} + (I - \widetilde{I})$$

where

$$\widetilde{I}(a,p) = \sum_{n=0}^{N} \frac{p_{\alpha_1}\ldots p_{\alpha_n}}{n!} \frac{\partial^n}{\partial p_{\alpha_1}\ldots\partial p_{\alpha_n}} I(a,p)\Big|_{p=0}$$

Order of expansion N determined by degree of UV divergences of $I(a,p)$ for $a \to 0$ ($N = 0$ – log divergence)
$\to$ $I - \widetilde{I}$ rendered UV finite and computed in continuum:

$$\left(I - \widetilde{I}\right)|_{a=0}$$

Taylor expansion of $\widetilde{I}(a,p)$ will in general create an IR divergence
Lattice integrals independent on external momentum p

To regularise the integrals dimensional regularisation with $D > 4$ is used:
IR poles of $\widetilde{I}$ cancel those of $I - \widetilde{I}$
in dim. regularisation $\widetilde{I}(0,p) = 0$ since the IR poles cancel exactly the UV poles
Consequence: UV divergent integral I transformed to an UV finite, but IR divergent expression

Remaining UV divergent contributions ($\propto 1/a^n$) of lattice integrals — if present — will cancel in operator representations

Rather cumbersome computations

$\rightarrow$ develop computer programs using algebraic manipulations
popular: FORM, MAPLE, MATHEMATICA

Some basic features of the developed program code:

- Package written in *MATHEMATICA*
- Symbolic Feynman rules used as input
  program computes one-loop forward matrix elements of bilinear quark on hyper-cubic lattice (including $O(a)$ contributions if needed) and in continuum in symbolic form
- Dimensional regularisation
- Symmetrisation tables used to accelerate momentum integration over Brillouin zone
- General index handling in the complicated case of hyper-cubic H(4) symmetry (non-Lorentz covariant structures)
- Algebraic isolation of IR poles which leads to an exact cancellation of divergences
- Results given with general index structure what allows an easy generation of all group representations
- All *finite* lattice integrals represented by symbols which are accurately calculated numerically

Example of complexity of calculation:
Number of initial terms in vertex contribution:
2 quark propagators: 2^2; 2 qqg- vertices: 20^2; gluon propagator: 8; vertex: 1
after a minor simplification: 9856 starting terms !!!
Symmetrisation with up to 12 indices
Huge intermediate expressions

Many checks:

computations in general covariant gauge:
cancellations of gauge-dependent parts between continuum and lattice results

Polarised operators with same numerical (!) Z's as corresponding unpolarised operators (strong check since analytic results are widely different)

No $1/a$ terms for overlap fermions because of chirality property

Gauge dependent parts independent on fermion propagators due to Ward-Takahashi identity (same contribution for Wilson and overlap fermions) – numerically checked having fermion propagators in individual contributions

Some results for renormalisation factors

Generalities

To relate numbers obtained from MC simulations to physical continuum quantities, a lattice renormalisation of relevant matrix elements is necessary
Connection given by

$$\langle O_i^{\mathrm{cont}} \rangle = \sum_j \left(\delta_{ij} - \frac{g^2}{16\pi^2} \left(R_{ij}^{\mathrm{lat}} - R_{ij}^{\mathrm{cont}} \right) \right) \langle O_j^{\mathrm{lat}} \rangle = \sum_j Z_{ij} \langle O_j^{\mathrm{lat}} \rangle$$

where

$$\langle O_i^{\text{cont,lat}} \rangle = \sum_j \left(\delta_{ij} + \frac{g^2}{16\pi^2} R_{ij}^{\text{cont,lat}} \right) \langle O_j^{\text{tree}} \rangle$$

are continuum and lattice 1-loop expressions, respectively
tree-level matrix element same in both cases

Differences $\Delta R_{ij} = R_{ij}^{\text{lat}} - R_{ij}^{\text{cont}}$ enter in renormalisation factors

$$Z_{ij}(a\mu, g) = \delta_{ij} - \frac{g^2}{16\pi^2} \Delta R_{ij}(a\mu)$$

which connect lattice to continuum

These Z factors are gauge-invariant, independent of the state, depend only on the scale $a\mu$

All numerical values presented are obtained for $r = 1$ and some selected ρ values

Quark self energy including tadpole contribution

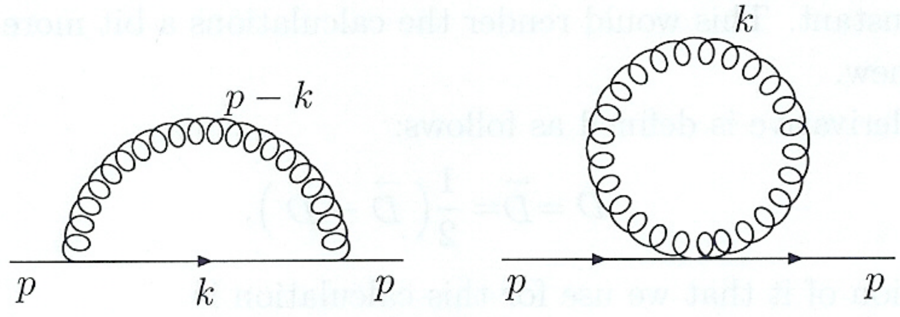

Figure 3: Quark self energy with tadpole contribution

Form of massless quark propagator

$$S^{-1} = \mathrm{i}\, \not{p} \left(1 - \frac{g^2 C_F}{16\pi^2} \Sigma_1 \right)$$

Lattice 1-loop correction:

$$\Sigma_1 = (1-\xi)\ \log(a^2p^2) + 4.79201\,\xi + b_\Sigma$$

ρ	b_Σ
0.6	-70.8443 - 402.384 C_1 - 235.659 C_2
1.0	-37.6306 - 198.881 C_1 - 117.143 C_2
1.4	-23.7660 - 112.041 C_1 - 66.7486 C_2

Table 1: Quark self energy contribution b_Σ; for $C_1 = C_2 = 0$ the results coincide with Alexandrou et al. and Capitani

Continuum 1-loop correction:

$$\Sigma_1^{\overline{MS}} = (1-\xi)\ \log\frac{p^2}{\mu^2} - 1 + \xi$$

Wave function

$$Z_\psi = 1 + \frac{g^2 C_F}{16\pi^2}\Sigma_1$$

$\rightarrow$ wave function renormalisation relating lattice and continuum quark propagator in $\overline{MS}$

$$Z_\psi^{\text{over}} \ = \ 1 + \frac{g^2 C_F}{16\pi^2}\left(2(1-\xi)\log(a\mu) + 3.79201\,\xi + b_\Sigma + 1\right)$$

$b_\Sigma + 1 = -36.6306 - 198.881\,C_1 - 117.143\,C_2$ for $\rho = r = 1$

Wilson fermions ($r = 1$)

$$Z_\psi^{\text{Wilson}} = 1 + \frac{g^2 C_F}{16\pi^2}\left(2(1-\xi)\log(a\mu) + 3.79201\,\xi + 12.8524\right)$$

Fermionic currents

Bilinear quark operators

$$\mathcal{O}_F = \bar{\psi}(x)\Gamma\psi(x)$$

Γ generic Dirac matrices

$$\Gamma^S = 1\,,\ \Gamma^P = \gamma_5\,,\ \Gamma^V = \gamma_\mu\,,\ \Gamma^A = \gamma_\mu\gamma_5\,,\ \Gamma^T = \sigma_{\mu\nu}\gamma_5$$

Amputated 1-loop contribution (without self energies)

$$\Lambda^{S,P} = \frac{g^2 C_F}{16\pi^2}\left[-(4-\xi)\log(a^2p^2) - 5.79201\xi + b_{S,P}\right]\{1,\gamma_5\}$$

$$\Lambda^{V,A}_\mu = \frac{g^2 C_F}{16\pi^2}\left[\gamma_\mu\left[-(1-\xi)\log(a^2p^2) - 4.79201\xi + b_{V,A}\right]\right.$$
$$\left. -2(1-\xi)\frac{p_\mu\,\not{p}}{p^2}\right]\{1,\gamma_5\}$$

$$\Lambda^T_{\mu\nu} = \frac{g^2 C_F}{16\pi^2}\sigma_{\mu\nu}\gamma_5\left[\xi\log(a^2p^2) - 3.79201\xi + b_T\right]$$

ρ	$b_{S,P}$
0.6	8.97983 + 6.32168 C_1 + 4.10601 C_2
1.0	10.2619 + 8.04619 C_1 + 5.24388 C_2
1.4	11.3434 + 10.0696 C_1 + 6.58738 C_2

ρ	$b_{V,A}$
0.6	6.32046 + 0.13056 C_1 + 0.100958 C_2
1.0	6.33125 + 0.17761 C_1 + 0.139203 C_2
1.4	6.34802 + 0.25233 C_1 + 0.201003 C_2

ρ	b_T
0.6	4.76733 - 1.93315 C_1 - 1.23406 C_2
1.0	4.35437 - 2.44525 C_1 - 1.56235 C_2
1.4	4.01624 - 3.02011 C_1 - 1.92779 C_2

Coefficients b_i coincide with Alexandrou et al. for plaquette action

Use known continuum 1-loop correction find gauge independent renormalisation factors

Example
Overlap fermions ($\rho = r = 1$)

$$Z_{S,P}^{\text{over}} = 1 - \frac{g^2 C_F}{16\pi^2}\Big(- 6\log(a\mu) - 32.3687 + 8.04619\, C_1 + 5.24388\, C_2 \Big)$$

Domain wall fermions, $C_1 = C_2 = 0$, domain wall height $M = 1$

$$Z_{S,P}^{\text{DWF}} = 1 - \frac{g^2 C_F}{16\pi^2}\Big(- 6\log(a\mu) - 17.5872(34) \Big)$$

Wilson fermions ($r = 1$)

$$Z_{S,P}^{\text{Wilson}} = 1 - \frac{g^2 C_F}{16\pi^2}\Big(- 6\log(a\mu) + \{12.95241, 22.59544\} \Big)$$

First moments of unpolarised and polarised non singlet quark distributions

Operators

$$\mathcal{O}_{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma_\mu \overleftrightarrow{D}_\nu\, \psi\,, \quad \mathcal{O}^5_{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma_\mu\gamma_5 \overleftrightarrow{D}_\nu\, \psi$$

$$\overrightarrow{D}_\mu\, \psi(x) = \frac{1}{2a}\Big(U_{x,\mu}\, \psi(x+\widehat{\mu}) - U^\dagger_{x-\widehat{\mu},\mu}\, \psi(x-\widehat{\mu}) \Big)$$

$$\bar{\psi}(x)\, \overleftarrow{D}_\mu = \frac{1}{2a}\Big(\bar{\psi}(x+\widehat{\mu})\, U^\dagger_{x,\mu} - \bar{\psi}(x-\widehat{\mu})\, U_{x-\widehat{\mu},\mu} \Big)$$

Consider two representations ($_{\{...\}}$ denotes complete symmetrisation)

$$\mathcal{O}_{v_2,d} = \mathcal{O}_{\{14\}}$$

$$\mathcal{O}_{v_2,e} = \mathcal{O}_{\{44\}} - \frac{1}{3}\left(\mathcal{O}_{\{11\}} + \mathcal{O}_{\{22\}} + \mathcal{O}_{\{33\}}\right)$$

find 1-loop contributions which coincide for $C_1 = C_2 = 0$ with Capitani

Same result for polarised case

Renormalisation factors

Overlap ($\rho = r = 1$)

$$\begin{aligned} Z^{\text{over}}_{v_2,d} &= 1 - \frac{g^2 C_F}{16\pi^2}\left(\frac{16}{3}\log(a\mu) - 48.8113 - 100.929\, C_1 - 59.2001\, C_2\right) \\ Z^{\text{over}}_{v_2,e} &= 1 - \frac{g^2 C_F}{16\pi^2}\left(\frac{16}{3}\log(a\mu) - 48.3773 - 100.514\, C_1 - 57.813\, C_2\right) \end{aligned}$$

Unpolarised Wilson:

$$\begin{aligned} Z^{\text{Wilson}}_{v_2,d} &= 1 - \frac{g^2 C_F}{16\pi^2}\left(\frac{16}{3}\log(a\mu) + 1.27959\right) \\ Z^{\text{Wilson}}_{v_2,e} &= 1 - \frac{g^2 C_F}{16\pi^2}\left(\frac{16}{3}\log(a\mu) + 2.56184\right) \end{aligned}$$

Polarised Wilson:

$$\begin{aligned} Z^{5,\text{Wilson}}_{v_2,d} &= 1 - \frac{g^2 C_F}{16\pi^2}\left(\frac{16}{3}\log(a\mu) + 0.34512\right) \\ Z^{5,\text{Wilson}}_{v_2,e} &= 1 - \frac{g^2 C_F}{16\pi^2}\left(\frac{16}{3}\log(a\mu) + 0.16738\right) \end{aligned}$$

Second moment in progress

Summary

- Computations performed in general covariant gauge using symbolic language *MATHEMATICA*
- Z factors found relating lattice and continuum operators of local fermionic currents and first moments of DI structure functions using overlap fermions and 6-link improved gauge actions
- Enormous computational effort
- Exact chiral symmetry maintained in all results
- Same Z factors for operators differing by γ_5
- Finite terms strongly differ from results of Wilson case

  Values of Z factors for overlap fermions are in general larger than for Wilson fermions

  trace largely to the quark self-energy and to the operators tadpoles

  Similar (but smaller) effect when improving Wilson fermions with a clover term

  Strong hope for better comparison with lattice measurements, add tadpole improvement

  First attempts with local scalar fermion current encouraging (QCDSF)
- Tendency makes perturbative calculations with improved actions more meaningful