List of problems 7

19. Find the shortest path (or geodesics) on a sphere of unit radius using spherical polar coordinates.
Hints: Use the polar angle $\theta$ as independent variable, and look for a path $\varphi = \varphi(\theta)$, where $\varphi$ is the azimuthal angle. To perform the integral, use the substitution $x = \cot \theta$.

20. Write down the equations of motion in polar coordinates for a particle of unit mass moving in a plane under a force with potential energy function $V(r, \theta) = -k \ln r + c r + g r \cos \theta$, where $k$, $c$, $g$ are positive constants. Find the positions of equilibrium (a) if $c > g$, and (b) if $c < g$.
By considering the equations of motion near these points, determine whether the equilibrium is stable (i.e. will the particle, if given a small displacement, tend to return?).

21. The energy and the generalized momenta of a mechanical system described by the Lagrangian $L(q, \dot{q}, t) \equiv L(q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s, t)$ are defined by
$$E = \sum_{i=1}^{s} p_i \dot{q}_i - L, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}.$$ Show that those quantities are transformed to
$$E' = E - \sum_{k=1}^{s} p_k \frac{\partial f_k}{\partial t}, \quad p'_i = \sum_{k=1}^{s} \frac{\partial f_k}{\partial q'_i} p_k$$ under the transformation of the generalized coordinates
$$q_i = f_i(q'_1, q'_2, \ldots, q'_s, t), \quad i = 1, \ldots, s.$$ Find the transformation laws for energy and generalized momenta corresponding to plane polar and cartesian coordinates for the coordinate transformations
(i) $\varphi = \varphi' + \Omega t, \quad r = r'$
(ii) $x = x' \cos \Omega t - y' \sin \Omega t, \quad y = x' \sin \Omega t + y' \cos \Omega t$
corresponding to a new coordinate system rotating with angular velocity $\Omega$ around the z axis.