Topological structure of the QCD vacuum seen by overlap fermions

Volker Weinberg
volker.weinberg@desy.de

NIC/DESY Zeuthen
Freie Universität Berlin

November 18, 2005

In collaboration with:

E.-M. Ilgenfritz, K. Koller, Y. Koma, G. Schierholz, T. Streuer

Outline

Outline

Motivation

Topological excitations, phenomenologically modeled as an interacting ensemble of instantons and anti-instantons, are believed to play a prominant role in the low energy behaviour of QCD:

- U(1)-anomaly, η' mass (Witten-Veneziano)
- θ-dependence
- chiral symmetry breaking; Banks-Casher-formula:
  $< \bar{\Psi}\Psi >= -\pi\rho(0)$

Motivation

- Goal: directly clarify from first principles whether the instanton picture realistically describes the QCD vaccum
- Overlap fermions are a appropriate tool to investigate the topological structure of QCD from first principles:
  - exact chiral symmetry on the lattice, given by the **Ginsparg-Wilson-Relation**: $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$ (P.H. Ginsparg and K.G. Wilson 1982)
  - index theorem exactly valid on the lattice

The overlap operator

The massive overlap operator

$$D = (1 - \frac{am_q}{2\rho})\, D_N + m_q,$$

$$D_N = \frac{\rho}{a}(1 + \frac{X}{\sqrt{X^\dagger X}}), X = D_W - \frac{\rho}{a},$$

- D_W Wilson-Dirac operator
- $0 \leq \rho \leq 2$ additional irrelevant parameter
- D_N has $n_- + n_+$ exact zero modes, n_- (n_+) being the number of modes with negative (positive) chirality.
- Index of D_N is thus given by $\nu = n_- - n_+$.
- 'Continuous' modes λ, $D_N\psi_\lambda = \lambda\psi_\lambda$, having $(\psi_\lambda^\dagger, \gamma_5\psi_\lambda) = 0$, come in complex conjugate pairs λ, λ^*.

Outline

Outline

T=0 simulation parameters

The Gauge action: quenched Luescher-Weisz

$$S[U] = \frac{6}{g^2}\Big[c_0 \sum_{\text{plaquette}} \frac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,(1 - U_{\text{plaquette}}) + c_1 \sum_{\text{rectangle}} \frac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,(1 - U_{\text{rectangle}}) + c_2 \sum_{\text{parallelogram}} \frac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,(1 - U_{\text{parallelogram}})\Big]$$

(coefficients c_1, c_2 ($c_0 + 8c_1 + 8c_2 = 1$) taken from tadpole improved perturbation theory.)

Fermions: overlap ($\rho = 1.4$)

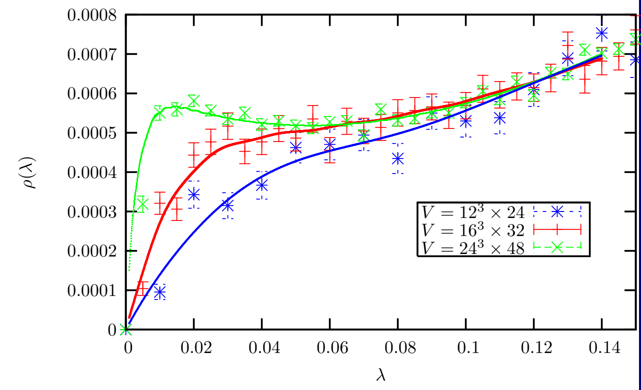
T=0 simulation parameters

Configurations

β	a [fm]	V (fm^4)	# of confs	# of modes
8.45	0.095	$12^3 \times 24$ (3.38)	500	O(50)
8.45	0.095	$16^3 \times 32$ (10.6)	267	O(140)
8.45	0.095	$24^3 \times 48$ (54.0)	186	O(160)
8.10	0.125	$12^3 \times 24$ (10.1)	254	O(140)

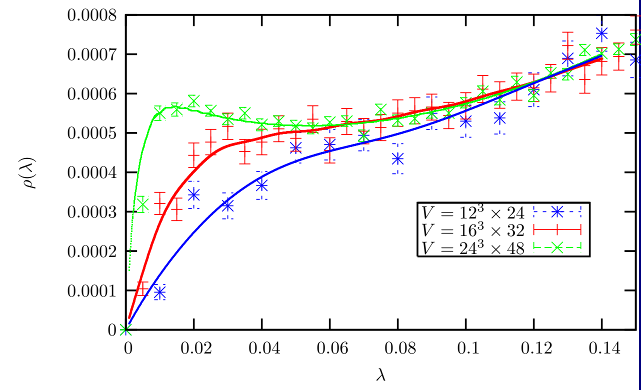
Outline

Volume dependence of the spectral density ($\beta = 8.45$)



$$\rho(\lambda) = V\Sigma_{eff}(\lambda) \sum_Q w(Q) \rho_Q(V\Sigma_{eff}(\lambda)\lambda)$$
$$\rho_Q(x) = \frac{x}{2}(J^2_{|Q|}(x) - J_{|Q|+1}(x) J_{|Q|-1}(x))$$

Volume dependence of the spectral density ($\beta = 8.45$)



From the fit:

$\delta = 0.20(3), \Sigma = (238(20)\mathrm{MeV})^3$

$\rightarrow \Sigma^{\overline{MS}}(2\mathrm{GeV}) = (258(21)\mathrm{MeV})^3 \; (Z_S = 1.27(1))$

The effective chiral condensate

Quenched Finite Volume Logarithms (Damgaard, 2001):

$$\Sigma_{eff}(V) = \Sigma(1 - 16\pi^2 \delta \partial_{M^2} \bar{\Delta}(M^2))$$

with $\delta = \frac{m_0^2}{48\pi^2 f^2}$ $\quad \bar{\Delta}(M^2) = \Delta(M^2) - \frac{1}{M^2 V}$ $\quad \Delta(M^2) = \frac{1}{V}\sum_p \frac{1}{p^2+M^2}$
Hasenfratz and Leutwyler (1990) :Propagator and any derivative thereof sparates into UV divergent + volume independent terms and additional finite volume-dependent terms.

$$\Delta(M^2) = \frac{M^2}{16\pi^2}(\ln(M^2) + c_1) + g_1(M^2, L_i)$$
$$\lim_{M^2 \to 0} \Sigma_{eff} = \Sigma(1 + 2\delta \ln(L) + 16\pi^2 \delta \beta_2(\tfrac{L_i}{L}))$$

$\beta_2(\frac{L_i}{L})$: ,,shape coefficient", $L = V^{1/4}$

$\rightarrow$ Logarithmic divergence of the effective chiral condensate

Outline

Connection to the Anderson transition

Spectral properties of the QCD Dirac operator resemble those of an disordered conductor at the Anderson transition:

solid state physis: conductivity produced by electrons that are initially bound to an impurity, but may become delocalised by overlapping with nearby impurities.

QCD: zero modes initially bound to a topological excitation, but may become delocalised by overlapping with other excitations.

Chracteristic critical features:

- eigenvectors multifractal
- spectral correlations well described by critical statistics

Localisation of eigenmodes

Inverse Participation Ratio I

$$I = V \sum_x \rho(x)^2$$

with the scalar density

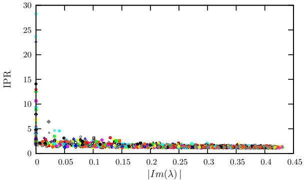
$$\rho(x) = {\Psi^\lambda}^\dagger(x)\Psi^\lambda(x)$$

using normalised eigenfunctions $\sum_x \rho(x) = 1$.

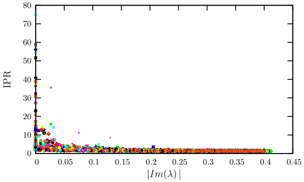
Characteristic Features:

$I = V$	localised	$\rho(x) = \delta_{x,x'}$	support only on one lattice point x'
$I = 1$	nonlocalised	$\rho(x) = \frac{1}{V}$	maximally spread on all sites
$I = \frac{1}{f}$			localised on fraction f of sites
$I = \frac{\pi}{2}$			Gaussian fluctuations

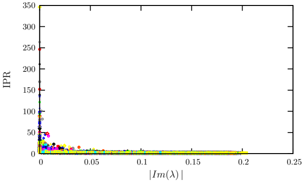
Localisation of eigenmodes



vol=$12^3 \times 24$

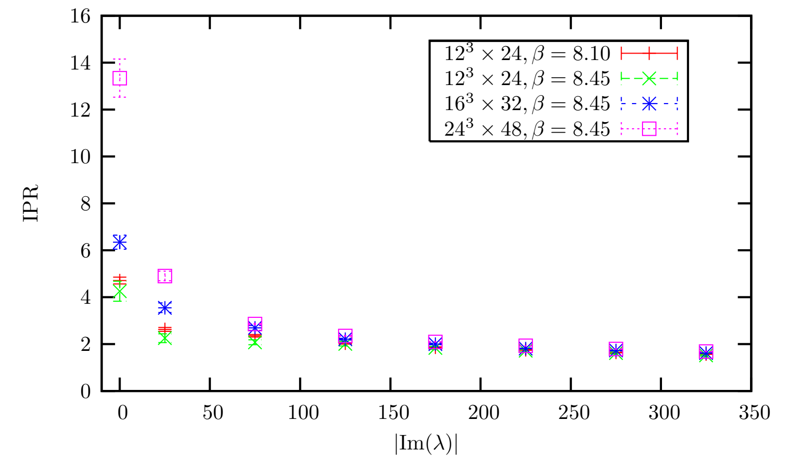


vol=$16^3 \times 32$



vol=$24^3 \times 48$

Lattice spacing and volume dependence of IPR



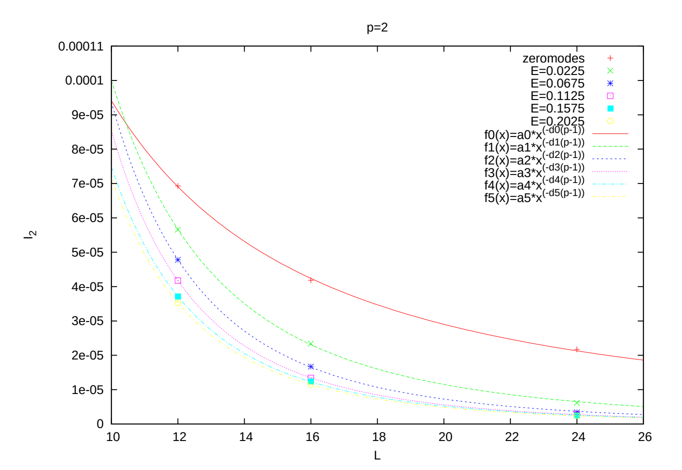
Multifractality and moments of inverse participation ratio

Moments of IPR

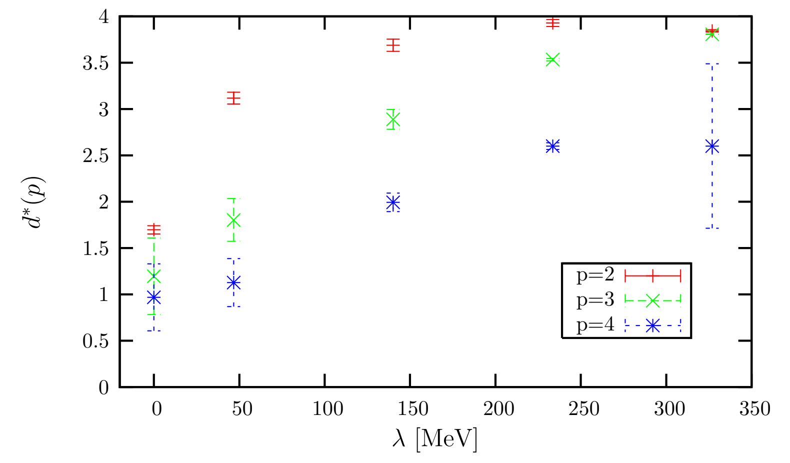
$$I_p(\lambda) = \sum_x \langle |\Psi_n(x)|^{2p} \delta(\lambda - \lambda_n) \rangle \propto \begin{cases} L^{-d(p-1)} & \text{metal} \\ L^{-d^*(p)(p-1)} & \text{critical} \\ const & \text{insulator} \end{cases}$$

$d^*(p) < d$ characterises the fractal dimensionality of the cluster where $|\Psi_n(x)|$ is larger than a certain value that increases with increasing p.

L-dependence of the moments of IPR (p=2)



$d^*(p)$ derived from the L-dependence of I_p



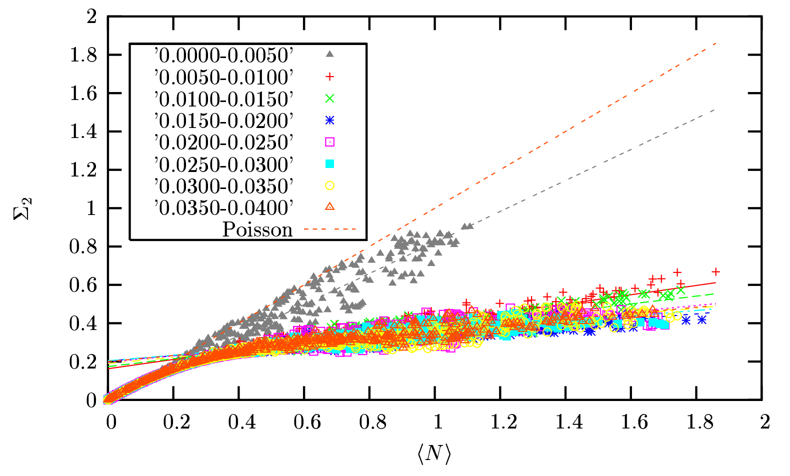
The level number variance $\Sigma_2(\langle N\rangle)$

Level number variance

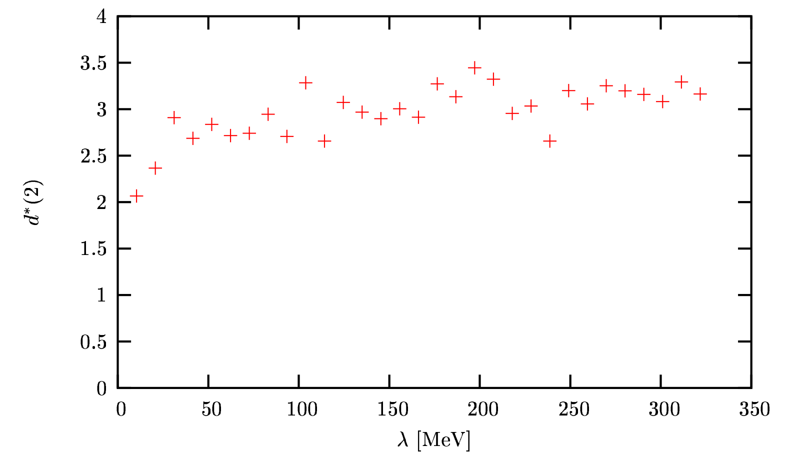
$$\Sigma_2(\langle N\rangle) = \langle (N - \langle N\rangle)^2\rangle \begin{cases} \propto \ln(\langle N\rangle) & \text{metal} \\ = \alpha\langle N\rangle & \text{critical} \\ = \langle N\rangle & \text{insulator} \end{cases}$$

level compressibility: $\alpha = \frac{d-d^*(2)}{2d}$

Level number variance



$d^*(2)$ derived from $\Sigma_2(\langle N \rangle)$



Outline

The topological charge density

The topological charge density for any γ_5-Hermitean Dirac operator satisfying the Ginsparg-Wilson relation is defined as (Hasenfratz, Laliena, Niedermayer 1998):

$$q(x) = \frac{1}{2}\,\mathrm{Tr}\,\gamma_5 D(x,x), \quad Q = \sum_x q(x)$$

Since $\mathrm{Tr}\,\gamma_5 = 0$ one can also write $q(x) = -\,\mathrm{Tr}\,\gamma_5(1 - \frac{1}{2}D(x,x))$

Using the spectral representation of the Dirac operator $D = \sum_\lambda \lambda\, |\Psi^\lambda >< \Psi^\lambda|$ the topological charge

$$q_\lambda(x) = -\sum_\lambda (1 - \frac{\lambda}{2}) c^\lambda(x), \quad c^\lambda(x) =< \Psi^\lambda(x)|\gamma_5|\Psi^\lambda(x) >$$

satisfies the index theorem for arbitrary truncation.

Methods of determining $q(x)$

- full charge density
  - we directly compute the trace of the Overlap-operator $\longrightarrow$ numerically very expensive, only done on 2-5 configurations
  - involves charge fluctuations at all scales, including physical QCD short-distance fluctuations and unphysical artifacts
- charge density based on $\mathcal{O}(150)$ low lying modes
  - low energy truncation, non-local representation of q(x)
  - filters out unphysical fluctuations at the scale of the cutoff $\longrightarrow$ UV-filter
  - gauge invariant approach, leaves the lattice scale unchanged in contrast to the cooling method

The local structure of the topological chage density

Visualisation of sign-coherent domains:

Isosurfaces of topological of topological charge density with $|q(x)|/q_{max} = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$, with red (green) surfaces indicating positive (negative) charges in 1 timeslice on 1 configuration

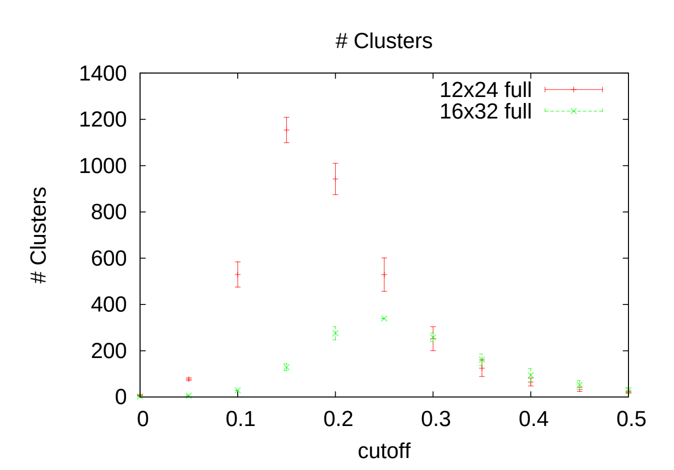
http://www.cip.physik.uni-muenchen.de/˜weinberg/topdens

Cluster analysis

- clusters formed by connected neighbours with $q(x) > \text{cutoff} = \{0.1, 0.2, 0.3, 0.4, 0.5\} * q_{max}$
- Analysis of:
  - number of clusters
  - fractional volume of clusters
  - total charge of the clusters
  - correlation function of the largest cluster
  - distance between the 2 leading clusters
  - cumulative clustercharge as a function of R

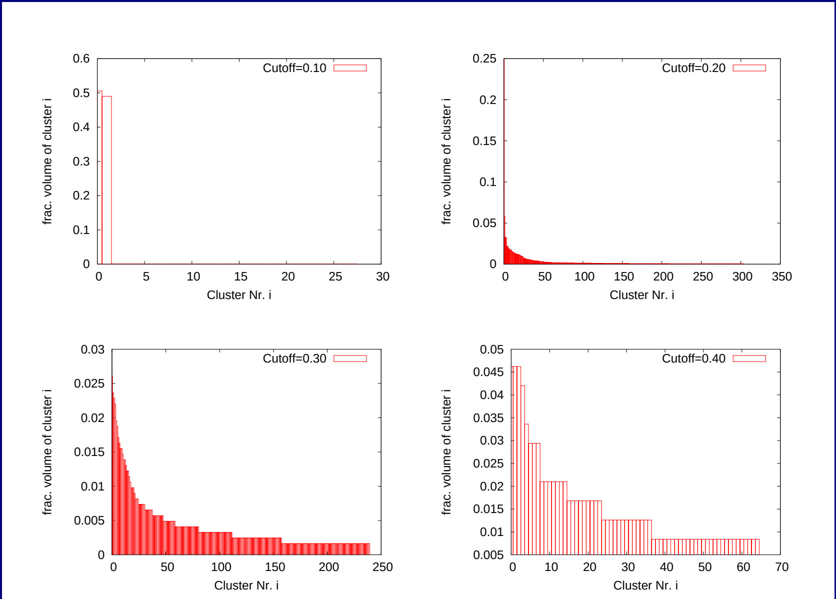
Data based on the eigenmode-expanded/full charge-density is based on $\mathcal{O}(100)/\mathcal{O}(3)$ configurations.

Number of clusters



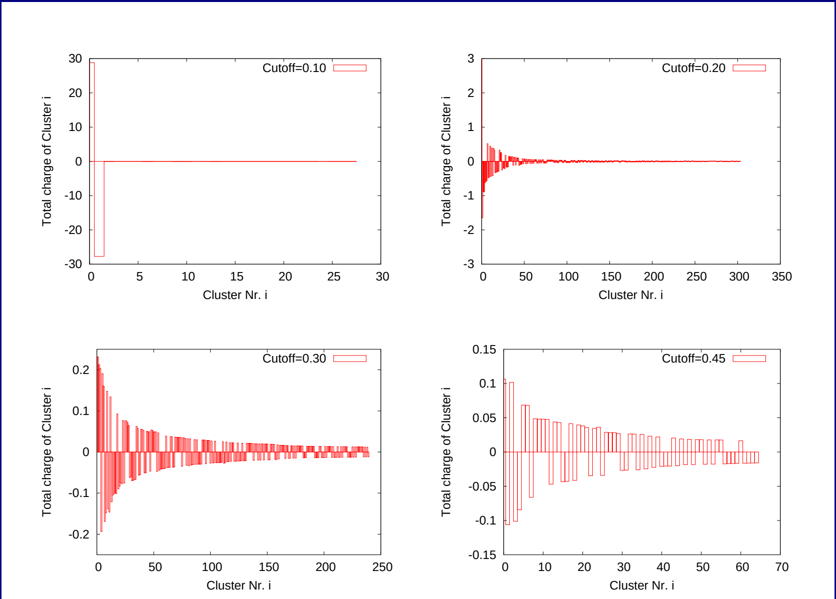
Fractional volume of the clusters

16 × 32, conf=03_0126, full charge density



Total charge of the clusters

16×32, conf=03_0126,$|Q| = 0$, full charge density

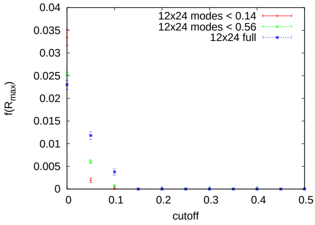


Correlation function of the largest cluster

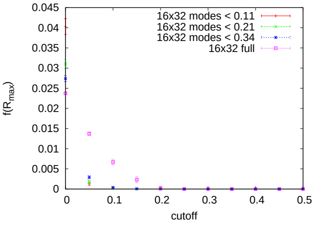
Cluster Correlation Function $f_c(R)$:

$$f_c(R) = \frac{\sum_{x_1} \sum_{x_2} \theta_c(x_1)\theta_c(x_2)\delta(|x_1 - x_2| - R))}{\sum_{x_1} \sum_{x_2} \delta(|x_1 - x_2| - R))}$$

Probability to be in the same cluster at distance R. Percolation iff $f(R_{max}) > 0$.



$12^3 \times 24$

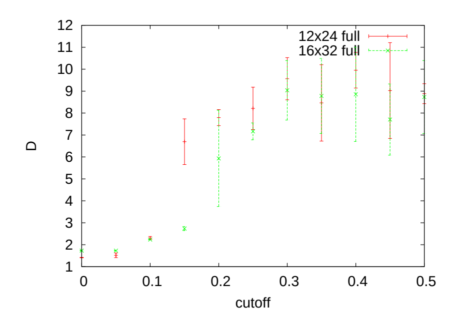


$16^3 \times 32$

Distance D between the 2 leading clusters

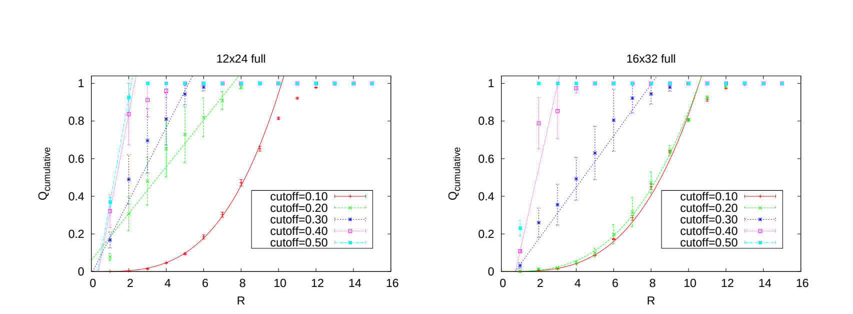
For each point *i* of a cluster calculate the distances to every point of the other cluster and take the minimum $distmin_i$ of these distances.

The distance D is defined as $max(distmin_i)$



Cumulative clustercharge as a function of the radius R

R=radius of a 4-D sphere around the center of the leading cluster
$Q_{cumulative}$=cumulative charge of the leading cluster with the given threshold inside the sphere relative to the full clustercharge



non-percolative regime: linear, percolative regime: power like ax^b

cutoff=0.10 → b=3.3(1)

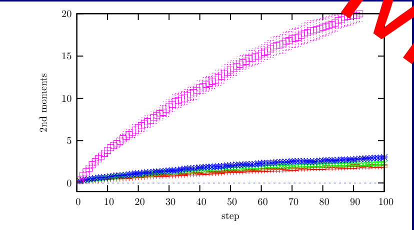
cutoff=0.10 → b=3.2(1)

cutoff=0.20 → b=2.9(1)

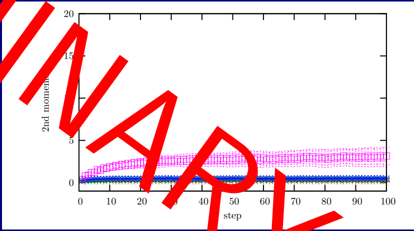
Exploring the largest cluster with random walkers

Distance traveled by 1000 random walkers in the directions of the principal axes reconstructed from the second moments

$$\begin{pmatrix} \langle xx \rangle & \langle xy \rangle & \langle xz \rangle & \langle xt \rangle \\ \langle yx \rangle & \langle yy \rangle & \langle yz \rangle & \langle yt \rangle \\ \langle zx \rangle & \langle zy \rangle & \langle zz \rangle & \langle zt \rangle \\ \langle tx \rangle & \langle ty \rangle & \langle tz \rangle & \langle tt \rangle \end{pmatrix} \xrightarrow{\text{principal axis transformation}} \begin{pmatrix} \langle \tilde{x}\tilde{x} \rangle & 0 & 0 & 0 \\ 0 & \langle \tilde{y}\tilde{y} \rangle & 0 & 0 \\ 0 & 0 & \langle \tilde{z}\tilde{z} \rangle & 0 \\ 0 & 0 & 0 & \langle \tilde{t}\tilde{t} \rangle \end{pmatrix}$$



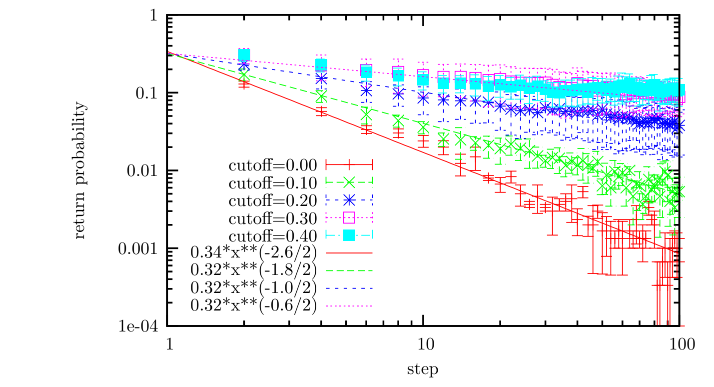
cutoff=0.10



cutoff=0.40

Return probability of random walkers

return probability $\propto$ step$^{-dim/2}$



Outline

Summary

- Since overlap fermions have exact zero modes and fulfill the index theorem exactly on the lattice, they offer the opportunity to probe the topological properties of gauge fields
- The spectral density is consistent with volume dependence predicted by quenched chiral perturbation theory
- Analysing the volume dependence of the IPRs of eigenmodes, we conclude that the zeromodes (low lying modes) extend in 2 (3) dimensions, while the higher modes seem to be objects extending in 4 dimensions.
- Performing a cluster analysis in dependence of the cutoff of the topological charge of the sites taken into account, we identify 1-D highly charged small laminar clusters on top of 2 dominating sign coherent percolating clusters of opposite charge, which are tangled and intertwined in a complex way.
- The QCD vacuum model with 4D coherent (anti)instantons with a typical instanton radius of $0.3 - 0.4$ fm is strongly modified by quantum fluctuations.

References I

- Y. Koma *et al.*,
Localization properties of the topological charge density and the low lying eigenmodes of overlap fermions,
PoS(LAT2005)300 [hep-lat/0509164].

- V. Weinberg *et. al.*,
Probing the chiral phase transition of n(f) = 2 clover fermions with valence overlap fermions,
PoS(LAT2005)171 [hep-lat/0510056].

- **QCDSF-UKQCD** Collaboration, D. Galletly *et al.*,
Quark spectra and light hadron phenomenology from overlap fermions with improved gauge field action,
Nucl. Phys. Proc. Suppl. **129** (2004) 453–455 [hep-lat/0310028].

References II

MILC Collaboration, C. Aubin *et al.*,
The scaling dimension of low lying Dirac eigenmodes and of the topological charge density,
[hep-lat/0410024].

P. Hasenfratz, V. Laliena and F. Niedermayer,
The index theorem in QCD with a finite cut-off,
Phys. Lett. **B427** (1998) 125–131 [hep-lat/9801021].

I. Horvath *et al.*,
On the local structure of topological charge fluctuations in QCD,
Phys. Rev. **D67** (2003) 011501 [hep-lat/0203027].

I. Horvath *et al.*,
Low-dimensional long-range topological charge structure in the QCD vacuum,
Phys. Rev. **D68** (2003) 114505 [hep-lat/0302009].

References III

- I. Horvath *et al.*,
  Inherently global nature of topological charge fluctuations in QCD,
  Phys. Lett. **B612** (2005) 21–28 [hep-lat/0501025].
- V. E. Kravtsov *et al.*,
  New Class of Random Matrix Ensembles with Multifractal Eigenvectors,
  [cond-mat/9703167].
- A. M. Garcia-Garcia and J. C. Osborn,
  Chiral phase transition as an Anderson transition in the instanton liquid model for QCD,
  [hep-lat/0509118]