Overlap Hypercube Fermions

in QCD with Light Quarks

I. **Construction of the overlap HF**

Locality, rotation symmetry and condition number

II. **Applications in the p-regime :**

m_q **vs.** m_π, m_ρ, $m_{\rm PCAC}$, Z_A **and** F_π

III. **Applications in the ϵ-regime :** F_π **and** Σ

Topological charges and susceptibility

Zero-mode contributions to $\langle PP \rangle$ in χ limit

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I. Construction of the Overlap HF

For free fermions, the **perfect lattice action** is known analytically (W.B./Wiese '95). Dirac operator:

$$D_{x,y} = \gamma_\mu \rho_\mu (x-y) + \lambda(x-y)$$

with closed expressions for $\rho_\mu(p), \ \lambda(p)$.

Based on iterated RG transformations $\to$ no lattice artifacts

Range of $D_{x,y}$ is infinite $\to$ *optimise* the RGT for *locality*, then truncate by periodic b.c. to a 3^4 hypercube
$\to \ \mathrm{supp}\, [\rho_\mu(x-y), \ \lambda(x-y)] \subset \{ |x_\nu - y_\nu| \leq 1 \}, \ \nu = 1 \dots 4$

"Hypercube Fermion" (HF), still excellent scaling (W.B./Brower/Chandrasekharan/Wiese, '96).

Gauging: sum over *shortest lattice paths* , plus **fat links**
$U_\mu(x) \to (1-\alpha)U_\mu(x) + \frac{\alpha}{6}\sum \text{staples}$; link amplification $\to$ criticality

Truncation and imperfect gauging $\to$ scaling and chirality somewhat distorted. Chirality can be corrected again by inserting the HF into the *overlap formula* (H. Neuberger, '97) :

$$D_{\rm ov} = \rho(1 + A/\sqrt{A^\dagger A}) \ , \quad A := D_0 - \rho \ , \quad \rho \gtrsim 1$$

where D_0 is some lattice Dirac operator (with $D_0 = \gamma_5 D_0^\dagger \gamma_5$).

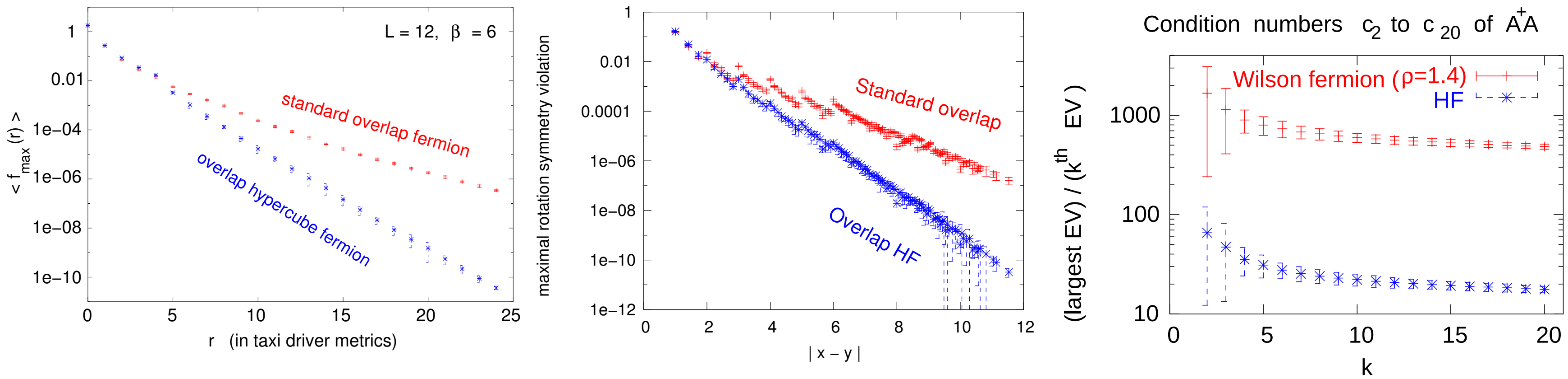
- Standard overlap fermion: based on $D_0 = D_{\rm Wilson}$
  Drastic change : $D_{\rm Wilson} \to D_{\rm ov-W}$
- Overlap HF: $D_0 = D_{\rm HF}$
  $D_{\rm HF}$ is approx. chiral already $\to$ modest modification : $D_{\rm HF} \to D_{\rm ov-HF}$

Both are Ginsparg-Wilson operators $\to$ *exact* χ *sym.* (P. Hasenfratz, '98, M. Lüscher, '98)

⇒ Virtues of the HF are essentially inherited by $D_{\mathrm{ov-HF}}$: high degree of

locality, approx. rotation symmetry, fast convergence

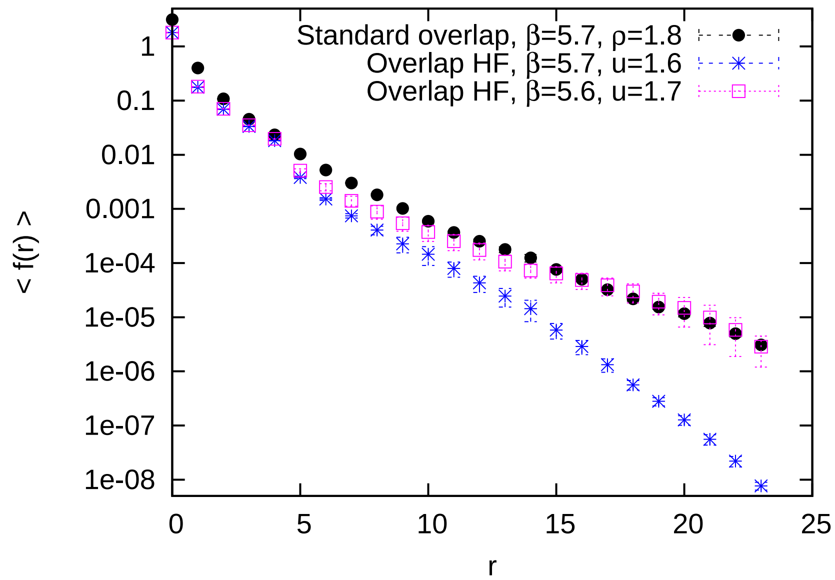
Examples at $\beta = 6$, $V = 12^4$, Wilson gauge action, quenched :



Overlap HF vs. standard overlap at $\rho = 1.4$

- $f_{\max}(r)$ "maximal correlation" between unit source $\bar{\psi}_x$ and ψ_y with $r = \|x-y\|_1$ (Hernández et al. '99)
- Measure for violation of rot. sym. : $f_{\max}(|x-y|) - f_{\min}(|x-y|)$
- Condition numbers c_k :
  (largest EV of $A^\dagger A$) / (smallest EV, after projecting out $k-1$ modes)
  Gain factor $\approx 25 \longrightarrow$ factor 5 in polynomial degree, vs. factor 15 of HF overhead

Locality at strong coupling:



- $\underline{\beta = 5.7}$ $(a \simeq 0.17 \text{ fm})$: $D_{\rm ov-W}$ with optimal ρ barely local
  $D_{\rm ov-HF}$ with optimal link amplification has powerful locality, stronger than $D_{\rm ov-W}$ at $\beta = 6$, $\rho = 1.4$
- $\underline{\beta = 5.6}$: for $D_{\rm ov-W}$ locality collapses, no valid Dirac operator
  but $\mathbf{D_{\rm ov-HF}}$ is still local

The overlap-HF formulation provides chiral fermions on coarser lattices.

II. Applications in the p-Regime

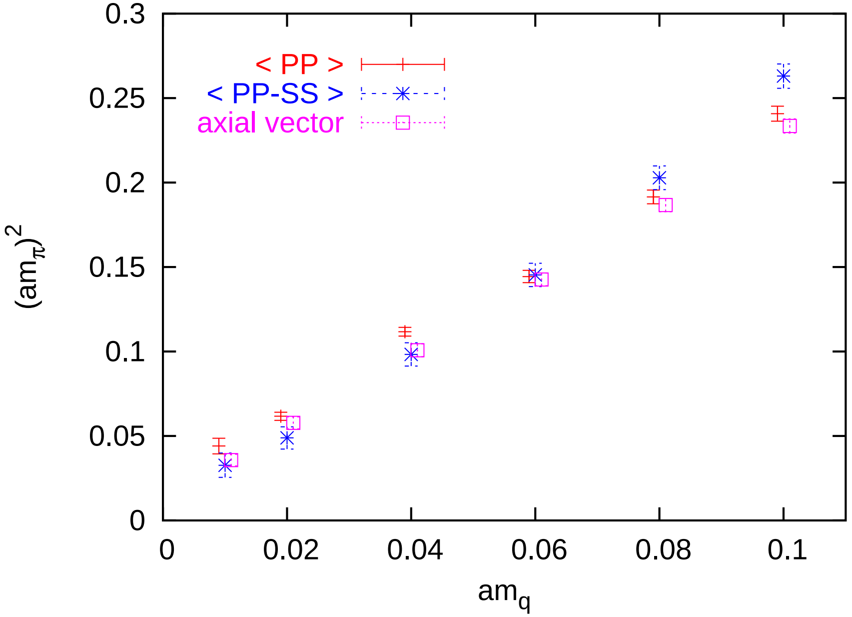
p-regime : p expansion of χPT is applicable, box length $\mathbf{L} \gg \frac{\mathbf{1}}{\mathbf{m}_\pi}$

We consider $\beta = 5.85 \quad (a \simeq 0.123 \text{ fm})$
$V = 12^3 \times 24 \quad , \qquad V_{\text{phys}} = (1.48 \text{ fm})^3 \times 2.96 \text{ fm}$
Bare quark masses : $am_q = 0.01,\ 0.02,\ 0.04,\ 0.06,\ 0.08,\ 0.1$
$\triangleq\ \mathbf{m_q \simeq 16.1}\ \text{MeV} \ldots \mathbf{161}\ \text{MeV} \ , \quad D_{\text{ov}}(m_q) = \left(1 - \frac{m_q}{2\rho}\right) D_{\text{ov}}^{(0)} + m_q$
{ *At smallest m_q close to ϵ-regime* }
Statistics : 100 propagators

We evaluate the **pion mass** in three ways:

- Pseudoscalar correlator $\langle PP \rangle$, $\quad P = \bar{\psi}\gamma_5\psi$
- $\langle A_4 A_4 \rangle$, $\quad A_4 = \bar{\psi}\gamma_5\gamma_4\psi$
- $\langle PP - SS \rangle$, $\quad S = \bar{\psi}\psi$: subtraction useful at small m_q,
  avoids contamination by zero modes, which plagues quenched results

Pion mass

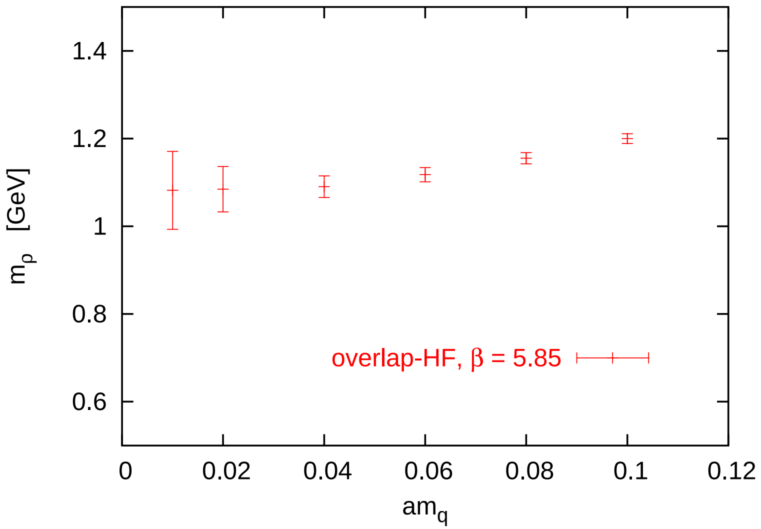


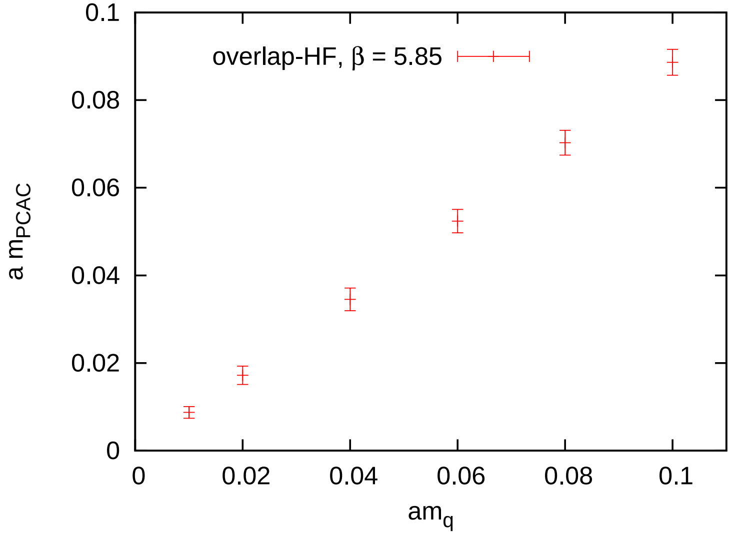
$m_\pi^2 \propto m_q$; Intercept for $m^2_{\pi,PP-SS}(m_q = 0) = -0.001(15)$

Hierarchy at small m_q : $m_{\pi,PP} > m_{\pi,AA} > m_{\pi,PP-SS}$
(agrees with P. Hasenfratz et al. '02)

Smallest pion mass : $m_{\pi,PP-SS}(am_q = 0.01) \simeq (279 \pm 32)$ MeV
$\longrightarrow \quad \frac{L}{\xi} \approx 2$, edge of p-regime

ρ-meson mass and quark mass according to the axial Ward identity:





Chiral extrapolation :

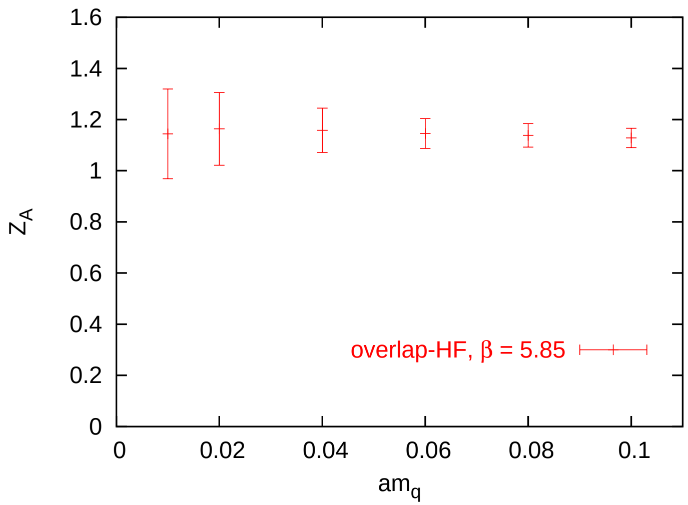
$m_\rho \in [978 \text{ MeV}, 1057 \text{ MeV}]$, $m_{\rm PCAC} \in [-0.00094, 0.00035]$

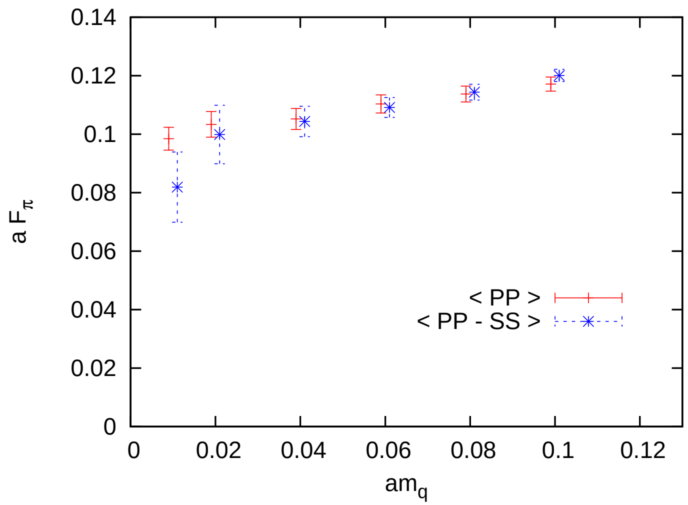
$$m_{\rm PCAC} = \sum_{\vec{x}} \langle \partial_4 A_4^\dagger(x) P(0) \rangle \, / \, 2 \sum_{\vec{x}} \langle P^\dagger(x) P(0) \rangle$$

(for $D_{\rm ov-W}$, see χLF, '04)

This yields the renormalisation constant $Z_A = \frac{m_q}{m_{\rm PCAC}}$,

and the pion decay constant $F_\pi = \frac{2m_q}{m_\pi^2} |\langle 0|P|\pi\rangle|$





Chiral extrapolation :

• $Z_A = 1.17(2)$; much closer to 1 than Z_A for standard overlap at same β
($Z_A \simeq 1.45$ at $\rho = 1.6$, χLF '04; $Z_A \simeq 1.55$ at $\beta = 6$, $\rho = 1.4$, Berruto et al. '03)

• $F_{\pi,PP} \in [109 \text{ MeV}, \ 114 \text{ MeV}]$, $F_{\pi,PP-SS} \in [95 \text{ MeV}, \ 113 \text{ MeV}]$
above theor. value in the χ limit: 86 MeV (Colangelo/Dürr, '04)
$\longrightarrow$ to be reconsidered in the ϵ-regime at smaller m_q . . .

III. Applications in the ϵ-Regime

ϵ-regime : $\frac{1}{m_\pi} > L$
In χPT p-expansion fails due to dominant 0-modes
But: analytical treatment of zero modes with collective variables (Gasser/Leutwyler, '87), higher modes captured by **ϵ-expansion**

Motivation: unphysical setting, but physical values of the **Low Energy Constants (LEC, free parameters in χPT, F_π, $\Sigma \dots$)** can be evaluated in small volumes
$\longrightarrow$ LEC at $V = \infty$.

Challenge for simulations with Ginsparg-Wilson fermions.
However: quenching $\longrightarrow$ log. finite size effects (Damgaard, '01)

Peculiarity of ϵ-regime: observables depend strongly on the *topological sector* (Leutwyler/Smilga, '92)

Evaluation of Σ from the Dirac Spectrum

χ **Random Matrix Theory** conjectures the densities of the **low lying Dirac eigenvalues** λ in the ϵ-regime (Damgaard/Nishigaki, '98) :

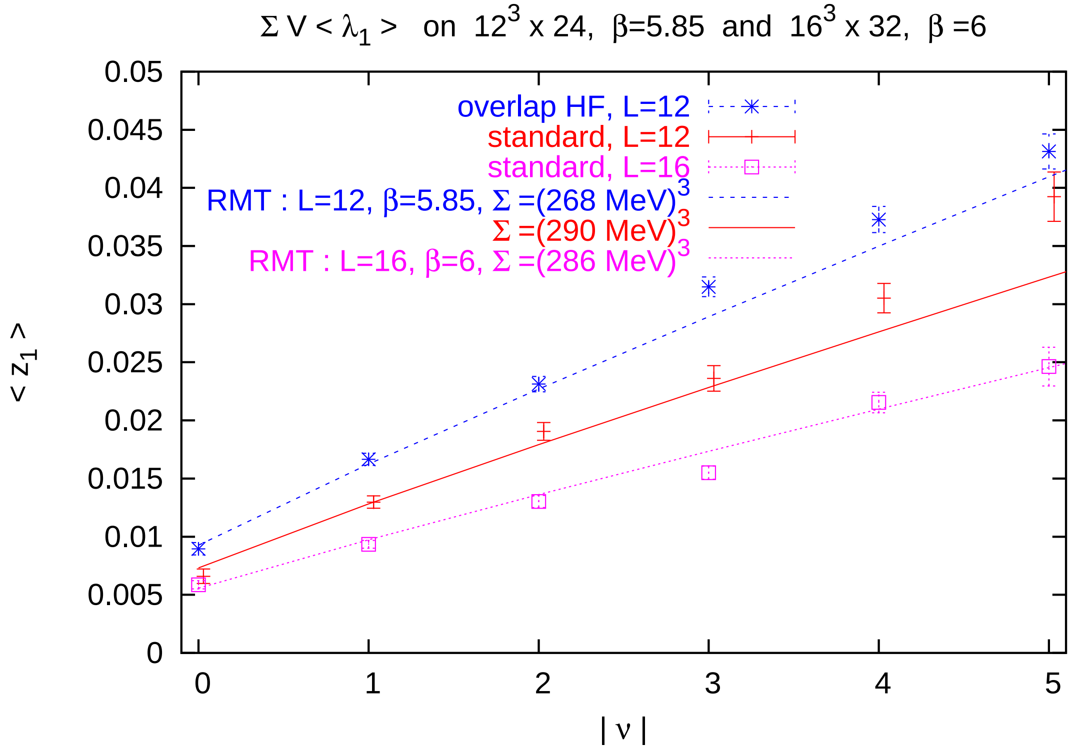
$$\rho_{\mathbf{n}}^{(\nu)}(\mathbf{z})$$

$z := \lambda \Sigma V$ (dim'less)
$n = 1, 2, 3 \dots$ (lowest eigenvalues, excluding zeros)
$\nu := \text{index} \equiv \text{top. charge}$

W.B./Jansen/Shcheredin, Giusti/Lüscher/Weisz/Wittig, Galletly et al., '03 :
χRMT predictions hold for leading EVs if the volume is not too small ($L \gtrsim 1.1 \text{ fm}$)

Scalar condensate Σ : only free parameter, determined by the fit.

$V = (1.48 \text{ fm})^3 \times 2.96 \text{ fm}$ at $a \simeq 0.12 \text{ fm}$ $(\beta = 5.85)$, and $a \simeq 0.093 \text{ fm}$ $(\beta = 6)$



Prediction works well in particular for $|\nu| = 0,\ 1,\ 2$

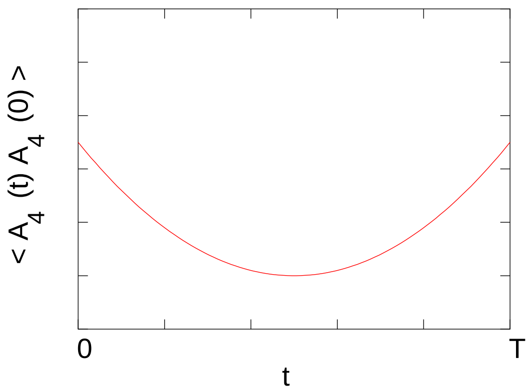
For the overlap-HF the fits yields : $\boldsymbol{\Sigma = (268(2)\ \text{MeV})^3}$

Preliminary result for F_π from the axial-vector correlator

Correlators to first order in **quenched** χPT (qχPT) (Damgaard et al., '02, '03)
$\langle Vector - Vector \rangle = 0$ (all orders)
$\langle Scalar - Scalar \rangle$ and $\langle Pseudoscalar - Pseudoscalar \rangle$
involve additional LEC, specific to quenching.

$\Rightarrow$ Focus on $\langle Axialvector - Axialvector \rangle$
in leading order **only** Σ **and** F_π

Bare axial current at $\vec{p} = \vec{0}$: $A_\mu(t) = \sum_{\vec{x}} \bar{\Psi}(t, \vec{x}) \gamma_5 \gamma_\mu \Psi(t, \vec{x})$
$\langle A_4(t) A_4(0) \rangle$ in a volume $L^3 \times T$:



Parabola: min. at $t = T/2$; F_π^2/T : additive, $\Sigma \longrightarrow$ curvature

W.B./Chiarappa/Jansen/Nagai/Shcheredin, '03 :
Simulations at $m_q = 21.3 \text{ MeV}$, $\beta = 6$ on $V = 10^3 \times 24$ and 12^4

- First lattice failed; consistent with χRMT study: $\mathbf{L \simeq 0.98 \text{ fm}}$ **is too small**
- MC history in $\nu = 0$ has **strong spikes** , smoother at $\nu \neq 0$,
  Spikes exactly for conf's with very **small EV** , most frequent at $\nu = 0$
  { Remedy: "Low Mode Averaging", Giusti et al. '04 }

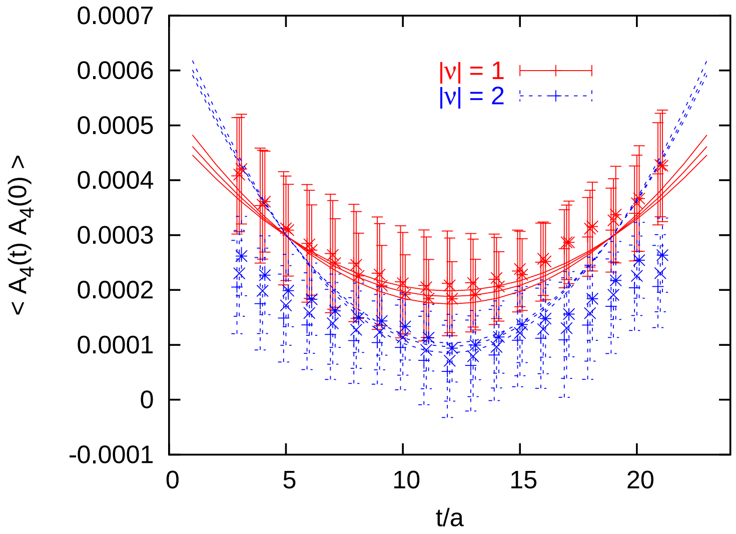
Decent agreement with qχPT in $V = (1.12 \text{ fm})^4$, $|\nu| = 1$:
$F_\pi = (86.7 \pm 4.0) \text{ MeV}$

Renormalisation with $Z_A \simeq 1.55 \longrightarrow$ $F_\pi^{\rm r} \approx 130 \text{ MeV}$ $> 93 \text{ MeV}$.

Σ cannot be extracted (curves hardly sensitive to $\Sigma = 0$ or $\Sigma = (250 \text{ MeV})^3$).

{ See also : Fukaya/Hashimoto/Ogawa, '05 }

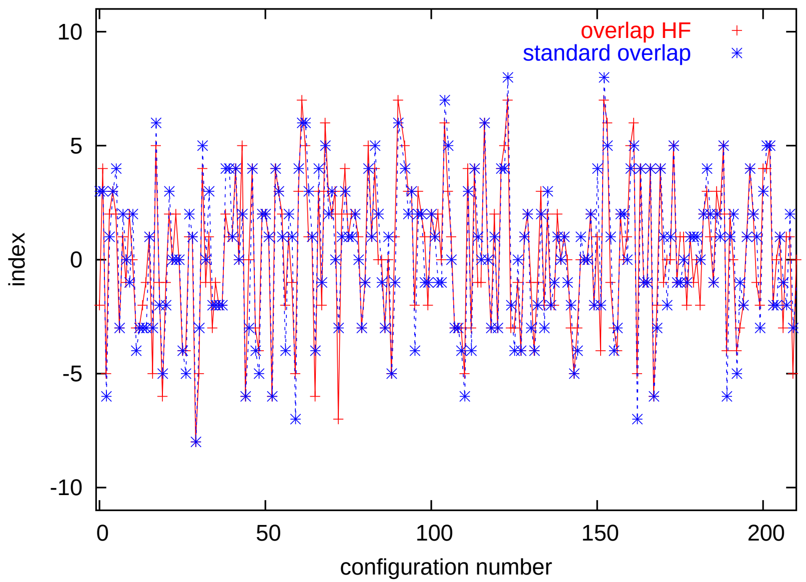
Preliminary result for overlap-HF $(12^3 \times 24,\ \beta = 5.85)$ 10 propagators in each of the sectors $|\nu| = 1$, $|\nu| = 2$; $\quad m_q = 1.6\ \mathrm{MeV},\ 4.8\ \mathrm{MeV},\ 8\ \mathrm{MeV}$

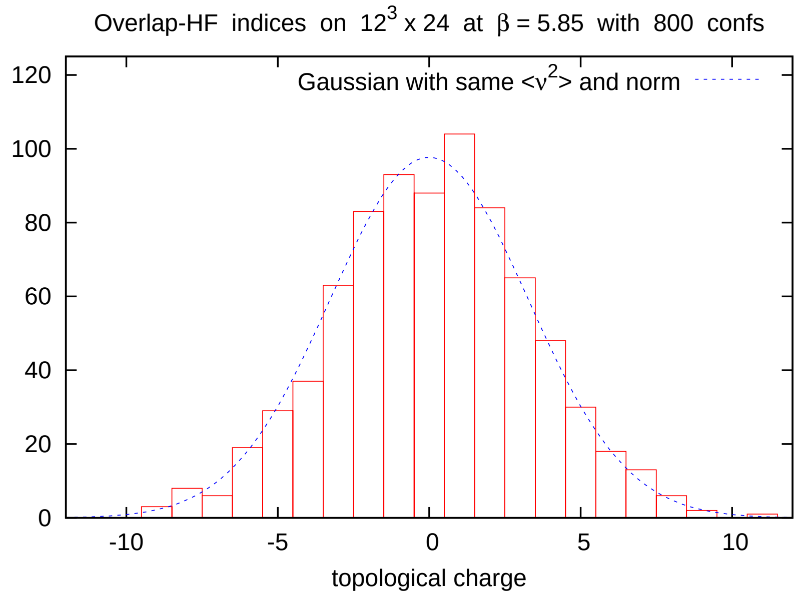


Global fit with $\Sigma = (268\ \mathrm{MeV})^3$: $F_\pi = (96 \pm 10)\ \mathrm{MeV}$
Renormalised : $F_\pi^{\mathrm{r}} = (104 \pm 9)\ \mathrm{MeV}$ (agrees with quenched literature)

Alternative method based on 0-mode contributions to meson correlators in the χ limit: see below.

Topological Charges :





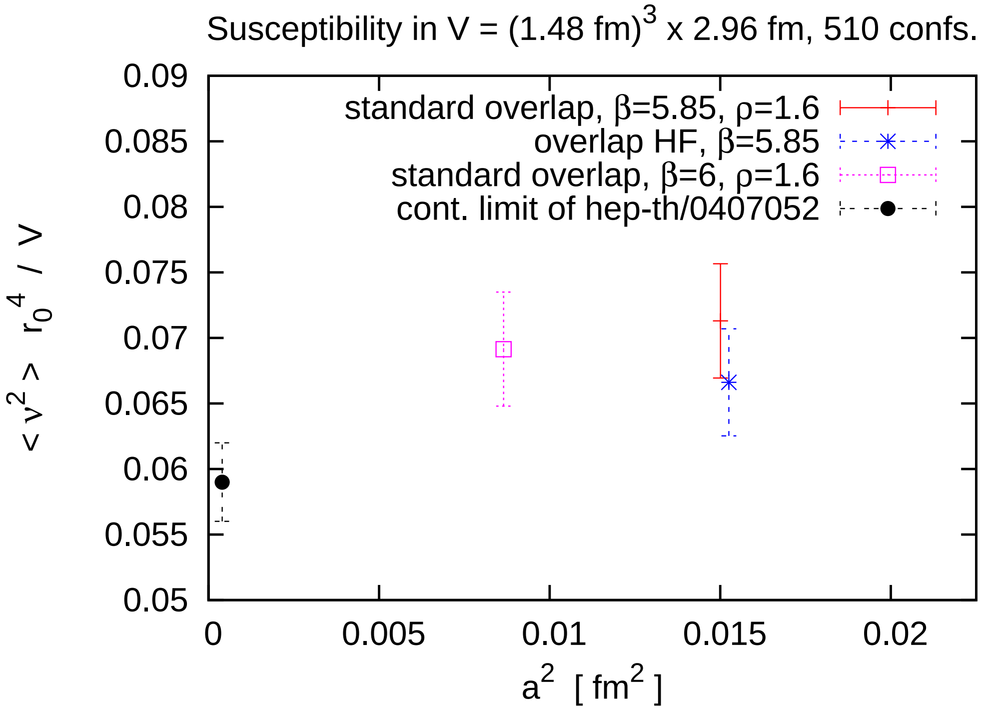
Left: part of index histories for standard overlap and overlap-HF, same conf's,

$\langle|\nu_{\mathrm{ov-W}} - \nu_{\mathrm{ov-HF}}|\rangle \approx 0.8$, max. deviation: 5

Right: charge histogram for overlap-HF, compatible with a Gaussian, parity (?)

(c.f. Alles/D'Elia/DiGiacomo, '05)

Topological susceptibility $\chi = \frac{1}{V}\langle\nu^2\rangle$ (relevant for mass of η')



Comparison to the continuum limit by Del Debbio/Giusti/Pica '04 (for $\langle\nu^2\rangle r_0^4/V$).

Considering finite a and $\rho = 1.6$: agreement within the errors

Overlap HF data closer to cont. limit (same confs at $\beta = 5.85$)

Giusti/Hernández/Laine/Weisz/Wittig '04: qχPT predictions for the

ZERO-MODE CONTRIBUTIONS to $\langle P\, P\rangle_{|\nu|}$

$\Rightarrow$ from the zero-modes alone we can evaluate F_π and α

α : Low Energy Constant, specific to quenching,
enters leading order of $\langle P\, P\rangle$.

$\mathcal{L}^{(2)}_{\mathrm{q}\chi\mathrm{PT}}$ is formulated with aux. scalar ϕ_0, which supplements the quenching effect
K : couples $\phi_0 U$, plus $\quad \frac{1}{2}\alpha_0(\partial_\mu\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2$
Count $m_0 = \mathcal{O}(\epsilon)$ and define $\alpha = \alpha_0 - 4N_c^2 K F_\pi/\Sigma$
$\Rightarrow$ first order observables only involve F_π and α (dim'less)

Pseudoscalar density $P = \bar{\psi}\gamma_5\psi$

$$
\begin{aligned}
\langle P(x)P(y)\rangle &= N_f P_1(x,y) - N_f^2 P_2(x,y) \\
P_1(x,y) &= \mathrm{Tr}[i\gamma_5(D+m_q)^{-1}(x,y)\cdot i\gamma_5(D+m_q)^{-1}(y,x)] \\
P_2(x,y) &= \mathrm{Tr}[i\gamma_5(D+m_q)^{-1}(x,x)]\cdot \mathrm{Tr}[i\gamma_5(D+m_q)^{-1}(y,y)]
\end{aligned}
$$

Spectral decomposition of propagators $\longrightarrow$ residuum given by zero modes:

$$\begin{aligned}
\lim_{m_q\to 0}(m_q V)^2\langle P(x)P(0)\rangle_\nu &= N_f C^{(1)}_{|\nu|}(x) + N_f^2 C^{(2)}_{|\nu|}(x) \\
\text{connected}: \quad C^{(1)}_{|\nu|}(x) &= -\langle v_j^\dagger(x)v_k(x)\cdot v_k^\dagger(0)v_j(0)\rangle_\nu \\
\text{disconnected}: \quad C^{(2)}_{|\nu|}(x) &= \langle v_j^\dagger(x)v_j(x)\cdot v_k^\dagger(0)v_k(0)\rangle_\nu
\end{aligned}$$

summed over (exact) **zero modes**, $Dv_j = 0$.

Spatial average: $\frac{1}{V}\int d^3x P(x)P(0) \quad \longrightarrow \quad C^{(i)}_{|\nu|}(t)$

Fits for $C^{(i)}_{|\nu|}(t)$ are troublesome; more handy:

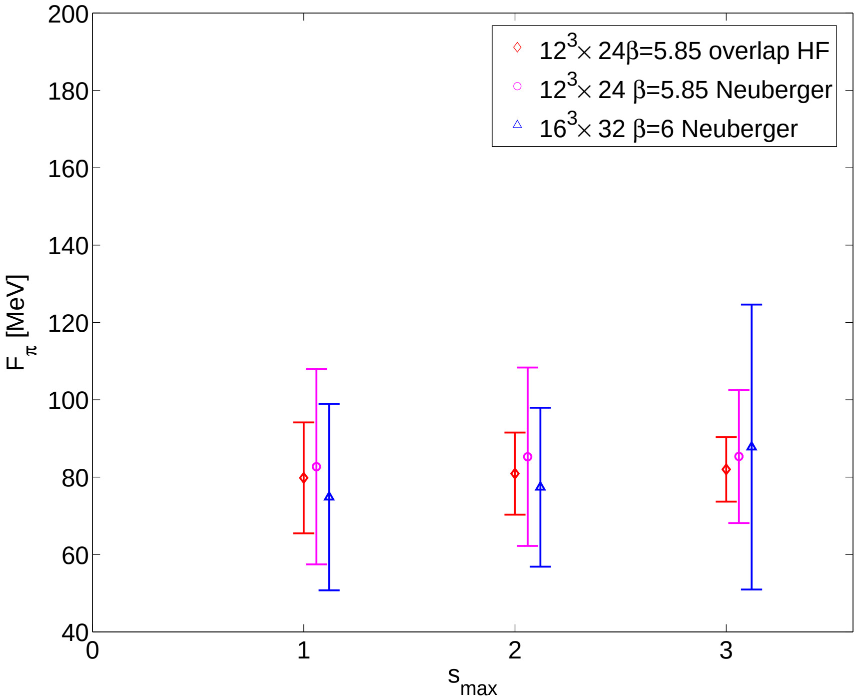
$$\frac{1}{L^2}\frac{d}{dt}C^{(i)}_{|\nu|}(t)|_{t=T/2} = D^{(i)}_{|\nu|}\cdot s + \mathcal{O}(s^3)\ , \qquad s = t - \frac{T}{2}$$

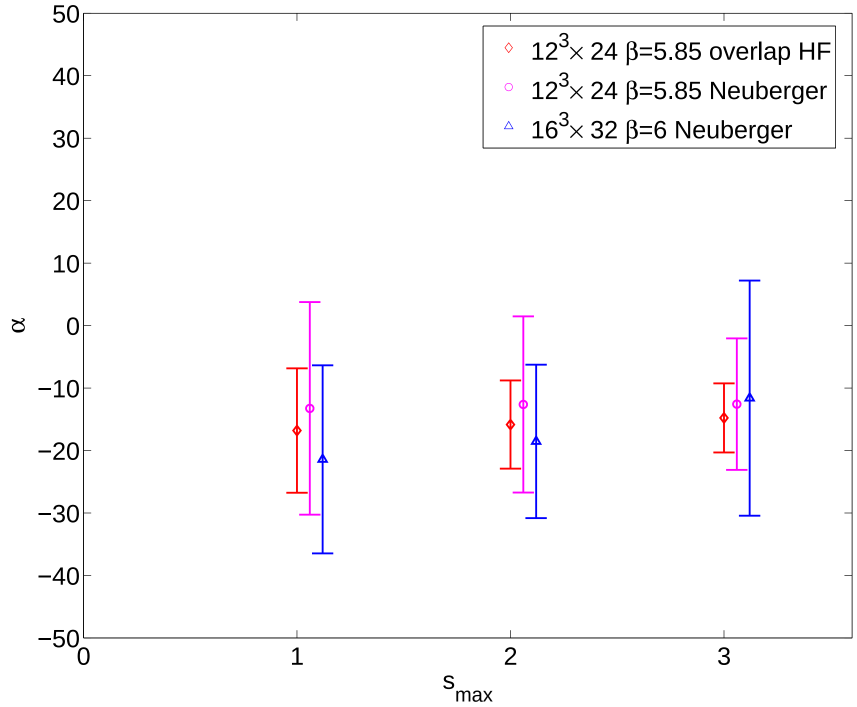
Measure slopes $D^{(i)}_{|\nu|}$ up to $s = \pm 1, \pm 2, \dots$, fits yield values for $\mathbf{F}_\pi,\ \alpha$

Combined fit for F_π and α for zero-mode pseudoscalar correlators of **standard overlap operator** and the **overlap-HF operator**
in $V = (1.48 \text{ fm})^3 \times 2.96 \text{ fm}$, $\langle \nu^2 \rangle$ as measured in each case.

Statistics :

Dirac operator	β	lattice size		$\vert\nu\vert = \mathbf{1}$	$\vert\nu\vert = \mathbf{2}$
standard overlap	5.85	$12^3 \times 24$		**132**	**115**
overlap HF	5.85	$12^3 \times 24$		**221**	**192**
standard overlap	6	$16^3 \times 32$		**115**	**94**





s : fitting range around $T/2$.

In particular the overlap HF yields a neat plateau.

$F_\pi = (80 \pm 14)$ MeV , $\alpha = -17 \pm 10$. Standard overlap data consistent.

F_π moves down compared to $\langle AA \rangle$ (theory: 86 MeV at $m_q = 0$).

Giusti et al. in $V \simeq (1.49 \text{ fm})^4 : F_\pi \simeq (117 \pm 16)$ MeV, $\alpha \in [-1.8, 7.8]$

Conclusions

- $D_{\mathrm{ov-HF}}$ provides **better locality** than $D_{\mathrm{ov-W}}$ $\longrightarrow$ applicable on **coarser lattices**
- **Rotation symmetry** improved
  Scaling promising (toy models, preliminary QCD results), under investigation
- p-regime : results for m_π, m_ρ, m_{PCAC}, F_π vs. m_q similar to standard overlap $\longrightarrow$
  independent confirmation, Z_A much closer to 1
- ϵ-regime : $\Sigma = (268(2)\ \mathrm{MeV})^3$ from Dirac spectrum
  $F_\pi \simeq 104(\pm 9)\ \mathrm{MeV}$ from $\langle A_4 A_4 \rangle$ *(preliminary)*

  agrees exactly with χ extrapolation from p-regime

  0-mode result at $m_q = 0$: $F_\pi = (80 \pm 14)\ \mathrm{MeV}$, $\alpha = -17 \pm 10$

  Topology conserving gauge action may be helpful

Thanks to M. Papinutto and C. Urbach for numerical tools

Modified lattice gauge action to "freeze" the topol. sector :

Hernández/Jansen/Lüscher '99, Neuberger '00

Theoretical condition for topology conservation:

$$
\begin{aligned}
S_P \quad &:= \quad 1 - \frac{1}{3}\mathrm{Re}\ \mathrm{Tr}(U_P) < \varepsilon \simeq 1/20.5 \\
U_P \quad &: \quad \text{any plaquette}
\end{aligned}
$$

Implementation (Lüscher '01, Fukaya/Onogi '03)

$$
S_\varepsilon(U_P) = \begin{cases} \frac{S_P(U_P)}{1 - S_P(U_P)/\varepsilon} & S_P(U_P) < \varepsilon \\ +\infty & \text{otherwise} \end{cases}
$$

W.B./Jansen/Nagai/Necco/Scorzato/Shcheredin '04

see also Fukaya/Hashimoto/Hirohashi/Ogawa/Onogi, '05

Relax ε, decrease β for $\approx$ const. scale, (local HMC)

$1/\varepsilon$	β	r_0/a	β_W	$\tau_{\text{aut}}^{\text{top}}$ / $\tau_{\text{aut}}^{\text{plaq}}$
0	6.18	7.14(3)	6.18	$:= \mathbf{1}$
1	1.5	6.6(2)	6.13	$\sim \mathbf{29}$
1.18	1	7.2(2)	6.18	$\sim \mathbf{80}$
1.52	0.3	7.3(4)	6.19	$\sim \mathbf{241}$

Approx. const. scale r_0/a on a 16^4 lattice
β_W : corresponding β value for the Wilson gauge action
τ_{aut} : autocorrelation time with respect to Q_{top} , and to the plaquettes

Allows us to sample a specific top. sector, desired in the ϵ-regime.

Also condition number for D_{ov} decreases.