UNIVERSITY OF LEIPZIG
INSTITUTE FOR THEORETICAL PHYSICS
Department: Theory of Elementary Particles

TP3 2017

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List of problems 11 (32. and 33. required, 34. voluntary)

32. Four identical coherent monochromatic wave sources A, B, C, D, located in the (x, y) plane, produce waves of the same wavelength λ. The source positions are given as follows:

$$A: \ (-\frac{\lambda}{2}, 0)\,, \quad B: \ (0, 0)\,, \quad C: \ (\frac{\lambda}{2}, 0)\,, \quad D: \ (0, \frac{\lambda}{2})\,.$$

Two receivers R_1 and R_2 in the same plane are at great (but equal) distances $r \gg \lambda$ from B at positions

$$R_1: \ (-r, 0)\,, \quad R_1: \ (0, -r)\,.$$

The intensity of a signal is $I \propto |\mathbf{E}|^2$.
Which receiver picks up the greater signal?
Which receiver, if any, picks up the greater signal if source B or source C are turned off?
Which receiver can tell which source, B or D, has been turned off?

33. A thin linear antenna of length d oriented in the z direction around the origin is excited in such a way that the sinusoidal current makes a full wavelength λ of oscillation ($k = 2\pi/d = 2\pi/\lambda = \omega/c$).

Note that the current flows in opposite directions in the top and the bottom half of this antenna. Thus the current density can be written as

$$\mathbf{J}(\mathbf{x}, t) = I_0 \sin(kz)\,\delta(x)\,\delta(y)\,\mathrm{e}^{-\mathrm{i}\,\omega t}\,\Theta(d/2 - |z|)\,\mathbf{e}_z$$

where $\Theta(\xi)$ is the unit step function [$\Theta(\xi) = 0(1)$ if $\xi < (>)0$].
Find the magnetic and electric fields in the radiation zone.
Calculate exactly the time-averaged radiated power per unit solid angle.
Hint: To calculate the radiation fields, start from the expression for the space-dependent part of the vector potential in the far zone ($\mathbf{n} = \mathbf{x}/r$)

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\,\frac{\mathrm{e}^{\mathrm{i}\,kr}}{r}\int \mathbf{J}(\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}\,k\mathbf{n}\cdot\mathbf{x}'}\,d^3x'\,.$$

34. voluntary, to collect two additional points

Starting from the general expression for the retarded scalar potential

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}$$

and the corresponding expression for $\mathbf{A}(\mathbf{x},t)$, expand both $|\mathbf{x}-\mathbf{x}'|$ and $t' = t - |\mathbf{x}-\mathbf{x}'|/c$ to first order in $|\mathbf{x}'|/r$ with $r = |\mathbf{x}|$ to obtain the electric dipole potentials for *arbitrary* time variation

$$\begin{aligned} \Phi(\mathbf{x},t) &= \frac{1}{4\pi\varepsilon_0} \left[\frac{1}{r^2} \mathbf{n}\cdot\mathbf{p}_{\mathrm{ret}} + \frac{1}{cr}\,\mathbf{n}\cdot\frac{\partial \mathbf{p}_{\mathrm{ret}}}{\partial t} \right], \\ \mathbf{A}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\,\frac{1}{r}\,\frac{\partial \mathbf{p}_{\mathrm{ret}}}{\partial t} \end{aligned}$$

where $\mathbf{p}_{\mathrm{ret}} = \mathbf{p}(t - r/c)$ is the dipole moment evaluated at the retarded time measured from the origin $t_{\mathrm{ret}} = t - r/c$ and $\mathbf{n} = \mathbf{x}/r$.

Hints: In deriving the expression for the scalar potential, drop the scalar Coulomb potential part which does not radiate.

For the vector potential only the lowest order term is needed. Relate the volume integral over the current density at the retarded time measured from the origin to $\dfrac{\partial \mathbf{p}_{\mathrm{ret}}}{\partial t}$.

(1 point)

Show that the fields *in the radiation zone* developed from this oscillating electric dipole $\mathbf{p}(t)$ can be written in the form

$$\mathbf{B}(\mathbf{x},t) = -\frac{\mu_0}{4\pi}\,\frac{1}{r\,c}\,\mathbf{n}\times\frac{\partial^2 \mathbf{p}_{\mathrm{ret}}}{\partial t^2}, \quad \mathbf{E} = -c\,\mathbf{n}\times\mathbf{B}(\mathbf{x},t)\,.$$

Show explicitly how you can go back and forth between these results and the harmonic fields used in the lecture by the substitutions

$$-\mathrm{i}\,\omega \leftrightarrow \frac{\partial}{\partial t} \quad \text{and} \quad \mathbf{p}\,\mathrm{e}^{\mathrm{i}\,kr - \mathrm{i}\,\omega t} \leftrightarrow \mathbf{p}_{\mathrm{ret}} = \mathbf{p}(t - r/c)\,.$$

(1 point)