UNIVERSITY OF LEIPZIG
INSTITUTE FOR THEORETICAL PHYSICS
Department: Theory of Elementary Particles

TP3 2017

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List of problems 8 (22. and 23. required, 24. voluntary)

22. Find the form of the Maxwell equations in vacuum (no medium, but sources present) under the transformation (θ is a constant angle)

$$\mathbf{E}' = \mathbf{E}\cos\theta + c\,\mathbf{B}\sin\theta\,,$$

$$\mathbf{B}' = -\frac{\mathbf{E}}{c}\sin\theta + \mathbf{B}\cos\theta\,.$$

Discuss the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ and the electromagnetic energy density $u = (1/2)\,(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$ under this transformation.

23. Starting with the retarded solution to the three-dimensional wave equation

$$\Psi(\mathbf{x},t) = \int \frac{f\left(\mathbf{x}', t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|}\, d^3x'\,,$$

show that the source $f(\mathbf{x}', t') = \delta(x')\,\delta(y')\,\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave,

$$\Psi(x,y,t) = \frac{2c\,\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function [$\Theta(\xi) = 0(1)$ if $\xi < (>) 0$].

24. voluntary, to collect an additional point:

Consider a point charge e moving along a prescribed path given by $\mathbf{r}(t)$ with arbitrary velocity $\mathbf{u}(t) = d\mathbf{r}/dt$. The corresponding sources are

$$\rho(\mathbf{x},t) = e\,\delta(\mathbf{x}-\mathbf{r}(t))\,,\quad \mathbf{J}(\mathbf{x},t) = \mathbf{u}(t)\,\rho(\mathbf{x},t)\,.$$

From the general expressions of the retarded electromagnetic potentials find for such a charge the Liénard-Wiechert potentials

$$\begin{aligned}
\Phi(\mathbf{x},t) &= \frac{e}{4\pi\varepsilon_0}\frac{1}{|\mathbf{R}(t_{\rm ret})| - \dfrac{\mathbf{u}(t_{\rm ret})\cdot\mathbf{R}(t_{\rm ret})}{c}}\,,\\
\mathbf{A}(\mathbf{x},t) &= \mathbf{u}(t_{\rm ret})\,\frac{\Phi(\mathbf{x},t)}{c^2}\,,\quad \mathbf{R}(t) = \mathbf{x}-\mathbf{r}(t)
\end{aligned}$$

where the retarded time is the solution of the relation

$$t_{\rm ret} + \frac{|\mathbf{R}(t_{\rm ret})|}{c} = t\,.$$

Note that both the instantaneous position and velocity of the point charge have to be taken at the retarded time.

Hints:

In the retarded potentials first reintroduce the integration over time t' via the δ function

$$\delta\left(t' - t + \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)$$

and than first integrate over $\mathbf{x}'$ using the three-dimensional δ function from the corresponding source.

The remaining δ function containing the time t' cannot be used explicitly for the integration over t'. Therefore, introduce use a new variable f

$$f = t' - t + \frac{|\mathbf{x} - \mathbf{r}(t'|}{c}$$

and use f for the integration. In other words, show that

$$\delta\left(t' - t + \frac{|\mathbf{x} - \mathbf{r}(t')|}{c}\right)\, dt' = \frac{\delta(f)}{\left(1 - \frac{\mathbf{u}\cdot(\mathbf{x}-\mathbf{r})}{c\,|\mathbf{x}-\mathbf{r}|}\right)\Big|_{f=0}}\, df$$

where the retarded time is the solution of $f = 0$.