UNIVERSITY OF LEIPZIG
INSTITUTE FOR THEORETICAL PHYSICS
Department: Theory of Elementary Particles

TP3 2017

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List of problems 7 (19. and 20. required, 21. voluntary)

19. Show that the function

$$S(q,\alpha,t) = \frac{1}{2} m\,\omega\,(q^2+\alpha^2)\,\cot\omega t - m\,\omega\,q\,\alpha\,\frac{1}{\sin\omega t}$$

is a solution of the Hamilton-Jacobi equation for the action function of a harmonic oscillator with the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2} m\,\omega^2 q^2\,.$$

Show that this solution generates a correct solution for the motion of the harmonic oscillator.

20. Suppose the potential in a problem of one degree of freedom is linearly dependent on time, such that Hamiltonian has the form

$$H(x,p,t) = \frac{p^2}{2m} - m\,A\,t\,x\,,$$

where A is a constant.
Solve the dynamical problem by means of Hamilton's action function (solution of the Hamilton-Jacobi equation (HJE)), under the initial conditions $t=0$, $x=0$, $p=mv_0$.
Result:

$$x(t) = v_0 t + \frac{A}{6} t^3\,, \quad p(t) = m\,\dot{x}(t)\,.$$

Hint: To construct the action function, postulate a solution of the form

$$S(x,t) = f(t)\,x + g(t)$$

and find the solutions for $f(t)$ and $g(t)$ by matching powers of x.

21. voluntary, to collect an additional point:
A canonical transformation is given by the generating function

$$\Phi(x, P) = xP + ax^3P + bxP^3 \,.$$

Choose the parameters a and b in such a way that small oscillations of an anharmonic oscillator $(m = 1)$ with

$$H = \frac{p^2}{2} + \frac{\omega_0^2 x^2}{2} + \beta x^4$$

$(\beta x^2 \ll \omega_0^2)$ become harmonic in the new variables Q and P. Neglect pieces of second order in $\beta\omega^{-2}Q^2$ in the new Hamilton function.
Hint: To find the solution after transforming to new variable Q, P, take into account that the solution for

$$H(x,p) = \frac{p^2}{2} + \frac{\omega_0^2\, x^2}{2} + \lambda \left(\frac{p^2}{2} + \frac{\omega_0^2\, x^2}{2} \right)^2$$

is

$$x(t) = A\cos(\omega t)\,, \quad p(t) = -\omega_0 A \sin(\omega t)\,, \quad \omega = \left(1 + \lambda\, A^2\, \omega_0^2\right) \omega_0 \,.$$

(Problem 17. of previous homework)
Result:

$$x(t) = Q - \frac{5\beta}{8\omega_0^2} Q^3 - \frac{9\beta}{8\omega_0^4} Q P^2$$

where

$$Q = A\cos(\omega t)\,, \quad P = -\omega_0 A \sin(\omega t)\,, \quad \omega = \omega_0 + \frac{3\beta}{2\omega_0} A^2 \,.$$