UNIVERSITY OF LEIPZIG
INSTITUTE FOR THEORETICAL PHYSICS
Department: Theory of Elementary Particles

TP3 2017

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List of problems 13 (Voluntary to get additional points)

38. (a) Find the magnitude of the momentum p of a relativistic particle of mass m as function of its kinetic energy T.
(b) A particle of mass m has the energy E. Determine its velocity v. Consider the non-relativistic and the extreme relativistic limits.
(c) Under the condition $v \ll c$ find approximate expressions for the kinetic energy T of a particle of mass m, expressed via its velocity v or momentum p to accuracy of v^4/c^4 or $p^4/(m^4 c^4)$, respectively.

39. Show that the annihilation of an electron-positron pair under emission of one real photon (mass zero) is forbidden by energy-momentum conservation, the emission of two photons is allowed.

40. A 4-vector is called timelike/spacelike if its invariant length squared (scalar product) is larger/smaller than zero in arbitrary inertial frames.
Show that the acceleration 4-vector defined as

$$(W^0, \mathbf{W}) = (\frac{dU^0}{d\tau}, \frac{d\mathbf{U}}{d\tau})$$

and expressed via the 3-velocity vector $\mathbf{u}$ and its derivative (3-acceleration vector) $\mathbf{a} = \dot{\mathbf{u}} \equiv \frac{d\mathbf{u}}{dt}$ is spacelike. Here τ is the proper time and $U^\alpha = (U^0, \mathbf{U})$ is the contravariant 4-velocity vector.

(a) $T = mc^2(\gamma_u - 1)$ (1) $\qquad \gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$ (2)

$\bar{p} = \gamma_u m \bar{u}$ (3)

from (1) $\quad \gamma_u = \frac{T + mc^2}{mc^2}$

from (2) $\quad \frac{1}{\gamma_u^2} = 1 - \frac{u^2}{c^2} \rightarrow \frac{u^2}{c^2} = 1 - \frac{1}{\gamma_u^2}$

$$\frac{u^2}{c^2} = 1 - \frac{(mc^2)^2}{(T+mc^2)^2} = \frac{(T+mc^2)^2 - (mc^2)^2}{(T+mc^2)^2} = \frac{T(T+2mc^2)}{(T+mc^2)^2}$$

from (3)

$$\bar{u}^2 = \frac{\bar{p}^2}{\gamma_u^2 m^2} \qquad \frac{\bar{u}^2}{c^2} = \frac{\bar{p}^2}{c^2 m^2 \gamma_u^2}$$

$$\Rightarrow \frac{\bar{p}^2}{c^2} = m^2 \gamma_u^2 \frac{\bar{u}^2}{c^2} = m^2 \frac{(T+mc^2)^2}{(mc^2)^2} \frac{T(T+2mc^2)}{(T+mc^2)^2}$$

$$\Rightarrow \bar{p}^2 = \frac{1}{c^2} T(T + 2mc^2)$$

$$\boxed{|\bar{p}| = \frac{1}{c}\sqrt{T(T+2mc^2)}}$$

(b) $p = \left(\frac{E}{c}, \bar{p}\right) = m\gamma_u (c, \bar{u})$

$$p \cdot p = p_0^2 - \bar{p}^2 = \frac{E^2}{c^2} - \bar{p}^2 = m^2 c^2$$

$$\Rightarrow \bar{p}^2 = \frac{E^2}{c^2} - m^2 c^2 \qquad ; \quad \bar{p} = \gamma_u m \bar{u}$$

$$\frac{\bar{u}^2}{c^2} = \frac{\bar{p}^2}{c^2} \frac{1}{m^2 \gamma_u^2} = \frac{\bar{p}^2}{c^2 m^2}\left(1 - \frac{\bar{u}^2}{c^2}\right)$$

$$\frac{\bar{u}^2}{c^2}\left(1 + \frac{\bar{p}^2}{c^2 m^2}\right) = \frac{\bar{p}^2}{c^2 m^2} \qquad \frac{u^2}{c^2} = \frac{\bar{p}^2}{\bar{p}^2 + c^2 m^2}$$

$$\frac{u^2}{c^2} = \frac{\frac{E^2}{c^2} - m^2 c^2}{E^2/c^2} = 1 - \left(\frac{mc^2}{E}\right)^2$$

$$\Rightarrow \beta = \frac{|\bar{u}|}{c} = \sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = \sqrt{1 - \left(\frac{E_0}{E}\right)^2}$$

$E_0 = mc^2$ rest energy

extrem relativistiz limit

$E_0 << E$

$$\beta \approx 1 - \frac{1}{2}\frac{E_0^2}{E^2}$$

Nonrelativistic limit $T << E_0$

$$E_{\text{nonrel}} = \frac{m}{2}\bar{u}^2 + mc^2 = T + mc^2 = T + E_0$$

$$\beta \approx \sqrt{1-\left(\frac{E_0}{T+E_0}\right)^2} = \sqrt{\frac{T^2+2TE_0+E_0^2-E_0^2}{(T+E_0)^2}} \approx \sqrt{\frac{2T}{E_0}}$$

check $\beta^2 = \frac{u^2}{c^2} = \frac{2T}{mc^2} \longrightarrow T = \frac{m}{2}u^2$

(c) $$T = mc^2\left(\frac{1}{\left(1-\frac{u^2}{c^2}\right)^{1/2}} - 1\right)$$

expansion $(1-x)^{-1/2} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2$

$$\Rightarrow \boxed{T \approx mc^2\left(\frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4}+\dots\right) = \frac{1}{2}mu^2 + \frac{3}{8}m\frac{u^4}{c^2}+\dots}$$

$$\bar{u}^2 = \frac{\bar{p}^2}{m^2\gamma_u^2} = \frac{\bar{p}^2}{m^2}\left(1-\frac{\bar{u}^2}{c^2}\right)$$

$$\Rightarrow \bar{u}^2\left(1+\frac{\bar{p}^2}{m^2c^2}\right) = \frac{\bar{p}^2}{m^2}$$

$$\bar{u}^2 = \frac{\bar{p}^2}{m^2}\frac{1}{1+\frac{\bar{p}^2}{m^2c^2}} \approx \frac{\bar{p}^2}{m^2}\left(1-\frac{\bar{p}^2}{m^2c^2}+\dots\right)$$

$\cancel{T \approx mc^2(}$

$$T = \frac{1}{2}m\frac{\bar{p}^2}{m^2}\left(1-\frac{\bar{p}^2}{m^2c^2}+\dots\right) + \frac{3}{8}m\frac{1}{c^2}\frac{\bar{p}^4}{m^4}+\dots$$

$$\boxed{T = \frac{\bar{p}^2}{2m} - \frac{1}{8}\frac{\bar{p}^4}{m^3c^2}+\dots}$$

$e^+ + e^- \to 1\gamma$

4-momentum conservation

$$p_{e^+} + p_{e^-} = p_\gamma$$

choose $c = 1$ (for simplicity)

photon massless

$$(p_{e^+} + p_{e^-})^2 = p_\gamma^2 \qquad p_e^2 = m_e^2 \quad p_\gamma^2 = 0$$

$$\downarrow$$

$$2m_e^2 + 2\, p_{e^+} \cdot p_{e^-} = 0$$

$\Rightarrow \; p_{e^+} \cdot p_{e^-} = -m_e^2 < 0 \quad (1)$ – all coord. frames

Use lab frame (e^- at rest)

$$p_{e^+} = (E_+, \bar{p}) \qquad p_{e^-} = (m_e, 0)$$

$$\Rightarrow \; p_{e^+} \cdot p_{e^-} = E_+ m_e > 0 \quad (2)$$

(2) in contradiction to (1)

$\Rightarrow$ kinematically ~~a decay~~ an annihilation of e^+e^- into real γ (mass = 0) is not allowed

$e^+ + e^- \to 2\gamma$

$$p_{e^+} + p_{e^-} = p_{\gamma 1} + p_{\gamma 2}$$

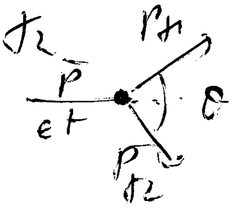
$$\hookrightarrow \; m_e^2 + p_{e^+} \cdot p_{e^-} = p_{\gamma 1} \cdot p_{\gamma 2}$$

lab frame:

$$m_e^2 + E_{e^+} m_e = E_{\gamma 1} E_{\gamma 2} - \underbrace{|\bar{p}_{\gamma 1}|}_{= E_{\gamma 1}} \underbrace{|\bar{p}_{\gamma 2}|}_{E_{\gamma 2}} \cos\theta$$

$$\Rightarrow m_e (m_e + E_{e^+}) = E_{\gamma 1} E_{\gamma 2} (1 - \cos\theta)$$

θ : angle of γ_1 with respect to γ_2



condition can be satisfied for certain energies of the photons

$\Rightarrow$ emission of 2γ kinematically allowed

$U^\alpha = (\gamma_u c, \gamma_u \bar{u})$

$W^\alpha = \frac{dU^\alpha}{d\tau} = \left(\frac{d}{d\tau}(\gamma_u c), \frac{d}{d\tau}(\gamma_u \bar{u})\right)$ $\qquad \gamma_u = \gamma_u(\bar{u}(t))$

~~Use~~ $W^0 = \frac{d}{d\tau}(\gamma_u c) = c\,\frac{d}{dt}(\gamma_u)\,\frac{dt}{d\tau}$; $\bar{a} = \frac{d\bar{u}}{dt}$

$\frac{dt}{d\tau} = \gamma_u \qquad \frac{d}{dt}\gamma_u = \frac{d}{dt}\left(1-\frac{u^2}{c^2}\right)^{-1/2} = -\frac{1}{2}\left(1-\frac{u^2}{c^2}\right)^{-3/2}(-)\frac{1}{c^2}\,2\bar{u}\cdot\bar{a}$

$= \frac{1}{c^2}\gamma_u^3\,\bar{u}\cdot\bar{a}$

$\boxed{W^0 = c\,\gamma_u \frac{1}{c^2}\gamma_u^3\,\bar{u}\cdot\bar{a} = \frac{1}{c}\gamma_u^4\,\bar{u}\cdot\bar{a}}$

$\bar{W} = \frac{d}{d\tau}(\gamma_u\bar{u}) = \frac{d}{dt}(\gamma_u\bar{u})\frac{dt}{d\tau} = \gamma_u\left(\bar{u}\frac{d}{dt}\gamma_u + \gamma_u\bar{a}\right)$

$= \gamma_u\left(\bar{u}\frac{1}{c^2}\gamma_u^3\,\bar{u}\cdot\bar{a} + \gamma_u\bar{a}\right) = \gamma_u^4\left(\frac{1}{c^2}(\bar{u}\cdot\bar{a})\bar{u} + \frac{1}{\gamma_u^2}\bar{a}\right)$

$= \gamma_u^4\left(\frac{1}{c^2}(\bar{u}\cdot\bar{a})\bar{u} + \left(1-\frac{\bar{u}^2}{c^2}\right)\bar{a}\right) = \gamma_u^4\left(\bar{a} + \frac{1}{c^2}\left((\bar{u}\cdot\bar{a})\bar{u} - \bar{u}^2\bar{a}\right)\right)$

$= \gamma_u^4\left(\bar{a} + \frac{1}{c^2}(\bar{u}\times(\bar{u}\times\bar{a}))\right)$

$\Rightarrow W^\alpha = \gamma_u^4\left(\frac{1}{c}\bar{u}\cdot\bar{a},\ \bar{a} + \frac{1}{c^2}(\bar{u}\times(\bar{u}\times\bar{a}))\right)$

introduce $\bar{\beta} = \frac{\bar{u}}{c}$

$\boxed{W^\alpha = \gamma_u^4\left(\bar{\beta}\cdot\bar{a},\ \bar{a} + \bar{\beta}\times(\bar{\beta}\times\bar{a})\right)}$ or

$\dot{\bar{\beta}} = \frac{1}{c}\bar{a}$

$\boxed{W^\alpha = \gamma_u^4 c\left(\bar{\beta}\cdot\dot{\bar{\beta}},\ \dot{\bar{\beta}} + \bar{\beta}\times(\bar{\beta}\times\dot{\bar{\beta}})\right)}$

Invariant

$W^2 = W\cdot W = W^\alpha W_\alpha = (W^0)^2 - \bar{W}^2$

$W^2 = \gamma_u^8 c^2 \underbrace{\left[(\bar{\beta}\cdot\dot{\bar{\beta}})^2 - \left(\dot{\bar{\beta}} + \bar{\beta}\times(\bar{\beta}\times\dot{\bar{\beta}})\right)^2\right]}_{I}$

$I = (\bar{\beta}\cdot\dot{\bar{\beta}})^2 - \dot{\bar{\beta}}^2 - 2\,\dot{\bar{\beta}}\cdot(\bar{\beta}\times(\bar{\beta}\times\dot{\bar{\beta}})) - \left(\bar{\beta}(\bar{\beta}\cdot\dot{\bar{\beta}}) - \dot{\bar{\beta}}\beta^2\right)^2$

$= (\bar{\beta}\cdot\dot{\bar{\beta}})^2 - \dot{\bar{\beta}}^2 - 2\dot{\bar{\beta}}\cdot\left(\bar{\beta}(\bar{\beta}\cdot\dot{\bar{\beta}}) - \dot{\bar{\beta}}\beta^2\right) - \beta^2(\bar{\beta}\cdot\dot{\bar{\beta}})^2 + 2(\bar{\beta}\cdot\dot{\bar{\beta}})^2\beta^2$

$- \dot{\bar{\beta}}^2\beta^4$

$= (\bar{\beta}\cdot\dot{\bar{\beta}})^2(1 - 2 - \beta^2 + 2\beta^2) - \dot{\bar{\beta}}^2(1 - 2\beta^2 + \beta^4)$

$= -\dot{\bar{\beta}}^2(1-\beta^2)^2 - (\bar{\beta}\cdot\dot{\bar{\beta}})^2(1-\beta^2)$

$$I = -\left(\frac{1}{\gamma_u^4}\dot{\vec\beta}^2 + \frac{1}{\gamma_u^2}(\vec\beta\cdot\dot{\vec\beta})^2\right)$$

$$\Rightarrow W^2 = \gamma_u^8 c^2 (-) \frac{1}{\gamma_u^4}\left(\dot{\vec\beta}^2 + \gamma_u^2(\vec\beta\cdot\dot{\vec\beta})^2\right)$$

$$\boxed{W^2 = -\gamma_u^4 c^2\left(\dot{\vec\beta}^2 + \gamma_u^2(\vec\beta\cdot\dot{\vec\beta})^2\right)}$$

$W^2 < 0$ Spacelike

alternative form

$$\dot{\vec\beta}^2 + \gamma_u^2(\vec\beta\cdot\dot{\vec\beta})^2 = \gamma_u^2\left[(1-\beta^2)\dot{\vec\beta}^2 + (\vec\beta\cdot\dot{\vec\beta})^2\right]$$

$$= \gamma_u^2\left[\dot{\vec\beta}^2 + (\vec\beta\cdot\dot{\vec\beta})^2 - \beta^2\dot{\vec\beta}^2\right] = \gamma_u^2\left[\dot{\vec\beta}^2 - (\vec\beta\times\dot{\vec\beta})^2\right]$$

$$\boxed{W^2 = -\gamma_u^6 c^2\left(\dot{\vec\beta}^2 - (\vec\beta\times\dot{\vec\beta})^2\right)}$$