UNIVERSITY OF LEIPZIG
INSTITUTE FOR THEORETICAL PHYSICS
Department: Theory of Elementary Particles

TP3 2017

Lecturer: PD Dr. A. Schiller

List of problems 12 (35. and 36. required, 37. voluntary)

35. Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \, .$$

Identify the corresponding Lorentz factor.
This is an alternative way to derive the parallel–velocity addition law.

36. A coordinate system K' moves with a velocity $\mathbf{v}$ relative to another system K. In K' a particle has a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the parallel and transverse components of the acceleration $\mathbf{a}_{\|}$ and $\mathbf{a}_{\perp}$ in system K with respect to the direction given by $\mathbf{v}$.

37. voluntary

Under a general Lorentz transformation with relative velocity $c\boldsymbol{\beta}$ between the inertial frames K and K' ($\gamma = 1/\sqrt{1-\beta^2}$) the electric and magnetic part of the electromagnetic field transforms as follows (later shown in the lecture)

$$\begin{aligned} \mathbf{E}' &= \gamma\left(\mathbf{E} + c\boldsymbol{\beta} \times \mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right) \, , \\ \mathbf{B}' &= \gamma\left(\mathbf{B} - \boldsymbol{\beta} \times \frac{\mathbf{E}}{c}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right) \, . \end{aligned}$$

Show that $\mathbf{E}^2 - c^2\,\mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$ are invariant under that Lorentz transformation.
Hint: It might be useful to decompose the electric and magnetic parts into longitudinal and transverse components with respect to $c\boldsymbol{\beta}$ and consider the transformations of those components individually.

$$x_0' = \gamma_1 (x_0 - \beta_1 x_1) \qquad x_1' = \gamma_1 (x_1 - \beta_1 x_0)$$

$$x_0'' = \gamma_2 (x_0' - \beta_2 x_1') \qquad x_1'' = \gamma_2 (x_1' - \beta_2 x_0')$$

$$x_0'' = \gamma_2 \left[\gamma_1 (x_0 - \beta_1 x_1) - \beta_2 \gamma_1 (x_1 - \beta_1 x_0) \right]$$

$$= \gamma_2 \gamma_1 \left[x_0 (1 + \beta_1 \beta_2) - x_1 (\beta_1 + \beta_2) \right]$$

$$= \underline{(\gamma_1 \gamma_2)} (1 + \beta_1 \beta_2) \left[x_0 - \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} x_1 \right]$$

$$x_1'' = \gamma_2 \left[\gamma_1 (x_1 - \beta_1 x_0) - \beta_2 \gamma_1 (x_0 - \beta_1 x_1) \right]$$

$$= \gamma_1 \gamma_2 \left[x_1 (1 + \beta_1 \beta_2) - x_0 (\beta_1 + \beta_2) \right]$$

$$= (\gamma_1 \gamma_2) (1 + \beta_1 \beta_2) \left[x_1 - x_0 \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \right]$$

Combined trafo

$$x_0'' = \gamma_{21} (x_0 - \beta_{21} x_1) \qquad x_1'' = \gamma_{21} (x_1 - \beta_{21} x_2)$$

$$\Rightarrow \beta_{21} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \quad \text{or} \quad v_{21} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

$$\gamma_{21} = \frac{1 + \beta_1 \beta_2}{\left[(1 - \beta_1^2)(1 - \beta_2^2) \right]^{1/2}}$$

<u>show</u> $\gamma_{21} = \dfrac{1}{\sqrt{1 - \beta_{21}^2}}$

$$\gamma_{21}^2 = \frac{1 + \beta_1^2 \beta_2^2 + 2 \beta_1 \beta_2}{(1 - \beta_1^2)(1 - \beta_2^2)}$$

$$\gamma_{21}^2 = \frac{1}{1 - \left(\frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \right)^2}$$

$$= \frac{(1 + \beta_1 \beta_2)^2}{(1 + \beta_1 \beta_2)^2 - (\beta_1 + \beta_2)^2}$$

$$= \frac{1 + \beta_1^2 \beta_2^2 + 2 \beta_1 \beta_2}{1 + \beta_1^2 \beta_2^2 + \cancel{2 \beta_1 \beta_2} - \beta_1^2 - \cancel{2 \beta_1 \beta_2} - \beta_2^2}$$

$$= \frac{1 + \beta_1^2 \beta_2^2 + 2 \beta_1 \beta_2}{(1 - \beta_1^2)(1 - \beta_2^2)} \quad \text{o.k.}$$

$$u_\parallel = \frac{u_\parallel' + v}{1 + \frac{v u_\parallel'}{c^2}} \qquad \bar{u}_\perp = \frac{\bar{u}_\perp'}{\gamma_v \left(1 + \frac{v u_\parallel'}{c^2}\right)} \qquad v u_\parallel' = \bar{v}\cdot\bar{u}$$

$$a_\parallel = \frac{du_\parallel}{dt} = \frac{du_\parallel}{dt'}\,\frac{dt'}{dt} \qquad \bar{a}_\perp = \frac{d\bar{u}_\perp}{dt} = \frac{d\bar{u}_\perp}{dt'}\,\frac{dt'}{dt}$$

find $$\frac{dt'}{dt} = \frac{dx_0'}{dx_0} = \gamma_v \frac{(dx_0 - \beta dx_1)}{dx_0} = \gamma_v \left(1 - \frac{v u_\parallel}{c^2}\right)$$

$$= \gamma_v \left(1 - \frac{1}{c^2} v \left(\frac{u_\parallel' + v}{1 + \frac{v u_\parallel'}{c^2}}\right)\right) = \frac{\gamma_v}{1 + \frac{v u_\parallel'}{c^2}} \left(1 + \cancel{\frac{v u_\parallel'}{c^2}} - \frac{v}{c^2}(\cancel{u_\parallel'} + v)\right)$$

$$= \frac{\gamma_v}{1 + \frac{v u_\parallel'}{c^2}} \left(1 - \frac{v^2}{c^2}\right) = \frac{1}{\gamma_v \left(1 + \frac{v u_\parallel'}{c^2}\right)} \equiv \frac{1}{\gamma_v \left(1 + \frac{\bar{v}\cdot\bar{u}'}{c^2}\right)}$$

$$\boxed{\frac{dt'}{dt} = \frac{1}{\gamma_v \left(1 + \frac{\bar{v}\cdot\bar{u}'}{c^2}\right)}}$$

$$\frac{du_\parallel}{dt'} = \frac{d}{dt'}\left(\frac{u_\parallel' + v}{1 + \frac{v u_\parallel'}{c^2}}\right) = \frac{a_\parallel'}{1 + \frac{v u_\parallel'}{c^2}} - \frac{(u_\parallel' + v)\left(\frac{v}{c^2} a_\parallel'\right)}{\left(1 + \frac{v u_\parallel'}{c^2}\right)^2}$$

$$= \frac{a_\parallel'}{\left(1 + \frac{v u_\parallel'}{c^2}\right)^2}\left(1 + \cancel{\frac{v u_\parallel'}{c^2}} - (\cancel{u_\parallel'} + v)\frac{v}{c^2}\right) = \frac{a_\parallel'}{\gamma_v^2 \left(1 + \frac{v u_\parallel'}{c^2}\right)^2}$$

$$\boxed{\frac{du_\parallel}{dt'} = \frac{a_\parallel'}{\gamma_v^2 \left(1 + \frac{\bar{v}\cdot\bar{u}'}{c^2}\right)^2}}$$

$$\frac{d\bar{u}_\perp}{dt'} = \frac{d}{dt'}\,\frac{\bar{u}_\perp'}{\gamma_v \left(1 + \frac{u_\parallel' v}{c^2}\right)} = \frac{1}{\gamma_v}\left\{\frac{\bar{a}_\perp'}{1 + \frac{u_\parallel' v}{c^2}} - \frac{\bar{u}_\perp'}{\left(1 + \frac{u_\parallel' v}{c^2}\right)^2}\frac{v}{c^2} a_\parallel'\right\}$$

$$= \frac{1}{\gamma_v \left(1 + \frac{u_\parallel' v}{c^2}\right)^2}\left\{\bar{a}_\perp' + \frac{\bar{a}_\perp' u_\parallel' v}{c^2} - \frac{\bar{u}_\perp' a_\parallel' v}{c^2}\right\}$$

Use $\bar{a}_\perp' = \bar{a}' - a_\parallel' \frac{\bar{v}}{v}$; $\bar{u}_\perp' = \bar{u}' - u_\parallel' \frac{\bar{v}}{v}$

$$\Rightarrow \bar{a}_\perp' + \frac{u_\parallel' v}{c^2}\left(\bar{a}' - \cancel{a_\parallel' \frac{\bar{v}}{v}}\right) - \frac{a_\parallel' v}{c^2}\left(\bar{u}' - \cancel{u_\parallel' \frac{\bar{v}}{v}}\right) = \bar{a}_\perp' + \frac{1}{c^2}\left(\bar{a}\,(\bar{u}'\cdot\bar{v}) - \bar{u}'\,(\bar{a}'\cdot\bar{v})\right)$$

$$= \bar{a}_\perp' + \frac{1}{c^2}\,\bar{v}\times(\bar{a}'\times\bar{u}')$$

$$\Rightarrow \frac{d\bar{u}_\perp}{dt'} = \frac{1}{\gamma_v \left(1 + \frac{\bar{u}'\cdot\bar{v}}{c^2}\right)^2}\left[\bar{a}_\perp' + \frac{1}{c^2}\,\bar{v}\times(\bar{a}'\times\bar{u}')\right]$$

$$\Rightarrow \left\| \vec{a}_\parallel = \frac{\bar{a}_\parallel'}{\gamma_v^3 \left(1 + \frac{\bar{v}\cdot\bar{u}'}{c^2}\right)^3} \qquad \bar{a}_\perp = \frac{\bar{a}_\perp' + \frac{1}{c^2}\,\bar{v}\times(\bar{a}'\times\bar{u}')}{\gamma_v^2 \left(1 + \frac{\bar{v}\cdot\bar{u}'}{c^2}\right)^3} \right\|$$

q.e.d.

$$\vec E' = \gamma(\vec E + c\vec\beta\times\vec B) - \frac{\gamma^2}{\gamma+1}\vec\beta(\vec\beta\cdot\vec E)$$

$$\vec E_\parallel = \frac{\vec\beta\cdot\vec E}{\beta^2}\vec\beta \qquad \vec E_\perp = \vec E - \vec E_\parallel$$

$$\vec B_\parallel = \frac{\vec\beta\cdot\vec B}{\beta^2}\vec\beta \qquad \vec B_\perp = \vec B - \vec B_\parallel \quad ; \quad \vec\beta\times\vec B_\parallel = 0$$

$$\Rightarrow \vec E' = \gamma(\vec E_\parallel + \vec E_\perp + \gamma c\vec\beta\times(\cancel{\vec B_\parallel} + \vec B_\perp)) - \frac{\gamma^2}{\gamma+1}\beta^2\vec E_\parallel$$

$$= \left(\gamma - \frac{\gamma^2\beta^2}{\gamma+1}\right)\vec E_\parallel + \gamma\vec E_\perp + c\vec\beta\times\vec B_\perp$$

$$\gamma - \frac{\gamma^2\beta^2}{\gamma+1} = \cancel{\gamma - \frac{\beta^2}{(\gamma+1)}}\frac{1}{(1-\beta^2)} \qquad \gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \beta^2 = \frac{\gamma^2-1}{\gamma^2}$$

$$= \gamma - \frac{\gamma^2-1}{1+\gamma} = \frac{\gamma+\gamma^2-\gamma^2+1}{1+\gamma} = 1$$

$$\Rightarrow \vec E' = \vec E_\parallel + \gamma\vec E_\perp + \gamma(c\vec\beta\times\vec B_\perp) \equiv \vec E'_\parallel + \vec E'_\perp$$

$$\Rightarrow \boxed{\vec E'_\parallel = \vec E_\parallel \qquad \vec E'_\perp = \gamma(\vec E_\perp + c\vec\beta\times\vec B_\perp)}$$

analogously for $\qquad \vec E \to \vec B c$, $\quad \vec B \to -\frac{\vec E}{c}$

$$\vec B' = \gamma\left(\vec B - \frac{\vec\beta}{c}\times\vec E\right) - \frac{\gamma^2}{\gamma+1}\vec\beta(\vec\beta\cdot\vec E)$$

$$\boxed{\vec B'_\parallel = \vec B_\parallel \qquad \vec B'_\perp = \gamma\left(\vec B_\perp - \vec\beta\times\frac{\vec E_\perp}{c}\right)}$$

$$\vec E'^2 - c^2\vec B'^2 = \vec E'^2_\parallel + \vec E'^2_\perp - c^2\vec B'^2_\parallel - c^2\vec B'^2_\perp = \vec E_\parallel^2 - \vec B_\parallel^2c^2 + \vec E'^2_\perp - c^2\vec B'^2_\perp$$

$$\vec E'^2_\perp - c^2\vec B'^2_\perp = \gamma^2\Big(\vec E_\perp^2 - c^2\vec B_\perp^2 + \cancel{2c\,\vec E_\perp\cdot(\vec\beta\times\vec B_\perp)} - \cancel{\tfrac{2}{c}\vec B_\perp\cdot(\vec\beta\times\vec E_\perp)c^2}$$

$$+ c^2(\vec\beta\times\vec B_\perp)^2 - \frac{1}{c^2}(\vec\beta\times\vec E_\perp)^2c^2\Big)$$

$$= \gamma^2\Big(\vec E_\perp^2 - c^2\vec B_\perp^2 + c^2(\beta^2\vec B_\perp^2 - \underbrace{(\vec\beta\cdot\vec E_\perp)^2}_{=0}) - \frac{c^2}{c^2}(\beta^2\vec E_\perp^2 - \underbrace{(\vec\beta\cdot\vec E_\perp)^2}_{=0})\Big)$$

$$= \gamma^2\left(\vec E_\perp^2(1-\beta^2) - c^2\vec B_\perp^2(1-\beta^2)\right) = (1-\beta^2)\gamma^2(\vec E_\perp^2 - c^2\vec B_\perp^2) = \vec E_\perp^2 - c^2\vec B_\perp^2$$

$$\Rightarrow \boxed{\vec E'^2 - c^2\vec B'^2 = \vec E^2 - c^2\vec B^2}$$

$$\vec E'\cdot\vec B' = (\vec E'_\parallel + \vec E'_\perp)\cdot(\vec B'_\parallel + \vec B'_\perp) = \vec E'_\parallel\cdot\vec B'_\parallel + \vec E'_\perp\cdot\vec B'_\perp = \vec E_\parallel\cdot\vec B_\parallel + \gamma^2\left[(\vec E_\perp + c\vec\beta\times\vec B_\perp)\cdot\left(\vec B_\perp - \vec\beta\times\frac{\vec E_\perp}{c}\right)\right]$$

$$= \vec E_\parallel\cdot\vec B_\parallel + \gamma^2\left[\vec E_\perp\cdot\vec B_\perp - (\vec\beta\times\vec B_\perp)\cdot(\vec\beta\times\vec E_\perp)\right]$$

$$= \vec E_\parallel\cdot\vec B_\parallel + \gamma^2\left[\vec E_\perp\cdot\vec B_\perp - \beta^2\vec E_\perp\cdot\vec B_\perp\right) \equiv \vec E_\parallel\cdot\vec B_\parallel + \vec E_\perp\cdot\vec B_\perp = \vec E\cdot\vec B$$

$$\Rightarrow \boxed{\vec E'\cdot\vec B' = \vec E\cdot\vec B}$$