The Temperature Zero Limit

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Abstract. We prove that, for a broad class of many–fermion models, the amplitudes of renormalized Feynman diagrams converge to their temperature zero values in the limit as the temperature tends to zero.

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§I. Formulation of the Model and Main Results

I.1 The Model

Consider a physical system, in dimension $d \geq 2$, consisting of a gas of fermions with prescribed density, possibly together with a crystal lattice of ions. If there is no lattice and there are no interactions between the fermions, the energy of one fermion, translated by the chemical potential μ, which controls the density of the gas, is $\frac{\mathbf{k}^2}{2m} - \mu$. If the fermions interact with each other through a two–body potential λu, the system has Hamiltonian

$$\int \tfrac{d^d\mathbf{k}}{(2\pi)^d}\, \big(\tfrac{\mathbf{k}^2}{2m} - \mu\big)\, a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} + \tfrac{1}{2} \int \prod_{i=1}^{4} \tfrac{d^d\mathbf{k}_i}{(2\pi)^d}\, (2\pi)^d\, \delta\big(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4\big)\, \lambda\, u\big(\mathbf{k}_1 - \mathbf{k}_3\big)\, a^\dagger_{\mathbf{k}_1,\sigma} a^\dagger_{\mathbf{k}_2,\tau} a_{\mathbf{k}_4,\tau} a_{\mathbf{k}_3,\sigma} \tag{I.1}$$

where the repeated spin indices σ and τ are summed over $\{\uparrow, \downarrow\}$ and $a_{\mathbf{k},\sigma}$ and $a^\dagger_{\mathbf{k},\sigma}$ annihilate and create, respectively, a fermion of momentum $\mathbf{k}$ and spin σ.

If there is a lattice, it provides a periodic background potential. Then, the energy of a single fermion, again in the absence of other interactions and again translated by the chemical potential, is called the dispersion relation and is denoted $E_n(\mathbf{k})$, where n is the band number and $\mathbf{k}$ is the crystal momentum. For notational simplicity, we restrict our attention to a single band and suppress the index n.

The grand canonical ensemble of these models, at inverse temperature $\beta = \frac{1}{kT}$ and chemical potential μ, is equivalently (formally) characterized by the Euclidean Green's functions

$$\Big\langle \prod_{i=1}^{N} \psi_{p_i,\sigma_i} \bar\psi_{q_i,\tau_i} \Big\rangle_\beta = \frac{\int \Big(\prod_{i=1}^{N} \psi_{p_i,\sigma_i} \bar\psi_{q_i,\tau_i}\Big)\, e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}} \tag{I.2a}$$

with action $\mathcal{A}$ and interaction $\mathcal{V}$ given by

$$\begin{aligned} \mathcal{A}(\psi,\bar\psi) &= -\int đk\ \big(ik_0 - E(\mathbf{k})\big) \bar\psi_{k,\sigma} \psi_{k,\sigma} - \mathcal{V}(\psi,\bar\psi) \\ \mathcal{V}(\psi,\bar\psi) &= \tfrac{\lambda}{2} \int \prod_{i=1}^{4} đk_i\ D\,(k_1 + k_2 - k_3 - k_4)\ \bar\psi_{k_1,\sigma} \psi_{k_3,\sigma} \langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle\, \bar\psi_{k_2,\tau} \psi_{k_4,\tau} \end{aligned} \tag{I.2b}$$

The "integral" notation is defined by

$$\int đk \equiv \frac{1}{\beta} \sum_{k_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int_{\mathcal{D}} \frac{d^d\mathbf{k}}{(2\pi)^d}$$

when $\beta < \infty$ and

$$\fint đk \equiv \int_{\mathbb{R}} \frac{dk_0}{2\pi} \int_{\mathcal{D}} \frac{d^d\mathbf{k}}{(2\pi)^d}$$

when $\beta = \infty$. The conservation of momentum "delta function" is defined by

$$D(k) = (2\pi)^d \beta\, \delta(\mathbf{k}) \delta_{k_0,0}$$

when $\beta < \infty$ and

$$D(k) = (2\pi)^{d+1} \delta(k)$$

when $\beta = \infty$. In the "momentum" $k = (k_0, \mathbf{k})$, the last d components $\mathbf{k}$ are to be thought of as a (crystal) momentum and the first component k_0 as the dual variable to a temperature, or to an imaginary time. We choose the set $\mathcal{D}$ of allowed spatial momenta to be some compact subset of $\mathbb{R}^d$, because at low temperature, the only important $\mathbf{k}$'s are those for which $E(\mathbf{k})$ is small. The fermion fields $\psi_{k,\sigma}, \bar{\psi}_{k,\sigma}$ are indexed by $k = (k_0, \mathbf{k}) \in \frac{\pi}{\beta}(2\mathbb{Z} + 1) \times \mathcal{D}$, $\sigma \in \{\uparrow, \downarrow\}$ and generate an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other:

$$\overset{(-)}{\psi}_{k,\sigma} \overset{(-)}{\psi}_{p,\tau} = -\overset{(-)}{\psi}_{p,\tau} \overset{(-)}{\psi}_{k,\sigma}$$

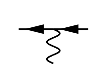
The interaction kernel that corresponds to the two–body potential u of (I.1) is

$$\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle \;=\; u(\mathbf{k}_1 - \mathbf{k}_3)$$

The fermions may also interact with lattice motion through the mediation of phonons. We allow for such interactions by allowing $\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle$ to be a general kernel. The precise hypotheses on these quantities and the precise mechanism for implementing the ultraviolet cutoff will be stated shortly.

§I.2 The Feynman Rules

If (I.2a) is Taylor expanded in powers of λ, the coefficient of λ^n is the sum of all Feynman diagrams of order n. The Feynman rules for these diagrams, when $\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle = u(\mathbf{k}_1 - \mathbf{k}_3)$, are

- Draw all topologically distinct connected graphs with $2n$ vertices , n interaction lines , $2n - N$ oriented internal particle lines and $2N$ oriented external particle lines.

- Assign a momentum to each line. Conserve momentum at each vertex. Once conservation of momentum has been incorporated, there are $n - N + 1$ *independent* internal momenta. They are summed/integrated using

$$\int đk \equiv \frac{1}{\beta} \sum_{k_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \frac{d^d\mathbf{k}}{(2\pi)^d}$$

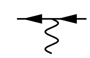
- Assign the propagator

$$\frac{\delta_{\sigma,\sigma'}}{ik_0 - E(\mathbf{k})}$$

  to each particle line and the interaction $\lambda \hat{u}(\mathbf{k})$ to each interaction line.
- Do the spin sums. This results in a factor of two for each fermion loop and a spin delta function for each fermion string.
- Multiply by $(-1)^n(-1)^F$ where F is the number of fermion loops.

Let G be any connected graph. It is both convenient and standard to get rid of the conservation of momentum delta functions arising in the value of G from the D in (I.2b) by integrating out some momenta. Then, instead of having one $(d+1)$–dimensional integration variable k for each line of the diagram, there is one for each momentum loop. Here is a convenient way to select these loops. Pick any spanning tree T for G that contains *all* the interaction lines. A spanning tree is a subgraph of G that is a tree and contains all the vertices of G. We associate to each line ℓ of $G \setminus T$ the "internal momentum loop" Λ_ℓ that consists of ℓ and the unique path in T joining the ends of ℓ. Fix any external line ℓ_{2N}. Associated to each external line $\ell_i,\ 1 \le i \le 2N-1$ other than ℓ_{2N} there is the "external momentum path" consisting of the unique path in T from ℓ_i to ℓ_{2N}. We say that the external momentum flows through every line ℓ on that path. The loop Λ_ℓ carries momentum k_ℓ. The component $\big(k_\ell\big)_0$ runs over $\frac{\pi}{\beta}(2\mathbb{Z}+1)$. The momentum $k_{\ell'}$ of each line $\ell' \in T$ is the signed sum of all loop and external momenta passing through ℓ'. If ℓ' is a particle line, the zero component of $k_{\ell'}$ is also required to lie in $\frac{\pi}{\beta}(2\mathbb{Z}+1)$. This is automatic by

Lemma I.1 *With the above choice of loops/paths (or any other choice of external momentum paths in T with the property that the number of external momentum paths ending at each* *external line is odd) the total number of loops/paths that traverse any particle line is odd and the total number of loops/paths that traverse any interaction line is even.*

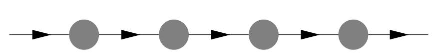
Proof: We identify any particle line proven to carry an odd number of loops/paths as well as any interaction line proven to carry an even number of loops/paths by painting it green. By hypothesis we may paint green all external lines. By construction, we may paint green all lines of $G \setminus T$, since they each carry exactly one loop. Furthermore, since any loop/path entering a vertex must also exit that vertex, once we know that any two lines of  are green, we may paint the third one green as well.

Pick any root for T. Start with the vertices farthest from the root in the partial ordering of T. Each of these farthest vertices has precisely one line in T. All the other lines of the vertex are either external or in $G \setminus T$ and hence are already green. So we may paint the one line that is in T green as well. Prune the vertex from the tree and repeat as required.

■

§I.3 Localization and renormalization

When $\beta = \infty$, many of the integrals generated by the Feynman rules of the last section are, in fact, not well–defined. We hasten to emphasize that this does **not** mean that the Euclidean Green's functions are ill–defined. It means that, with the dependence on λ specified in (I.2), the Euclidean Green's functions are not C^∞ in λ. The source of the difficulty is the singularity of the propagator $\frac{1}{ik_0 - E(\mathbf{k})}$. For example, when $E(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$, $\frac{1}{ik_0 - E(\mathbf{k})}$ has singular locus $k_0 = 0$, $|\mathbf{k}| = \sqrt{2m\mu}$. This propagator is locally L^1 but not locally L^p for any $p \geq 2$. It is very easy to get $\frac{1}{(ik_0 - E(\mathbf{k}))^p}$ with $p \geq 2$ arising in a Feynman diagram. It suffices for the diagram to contain a string



One may prevent strings from arising by reorganizing the perturbation expansion as a sum of skeleton graphs. By definition, a skeleton graph is one which contains no nontrivial strings. Restricting to skeleton graphs really is just a reorganization of the perturbation expansion, if we take for the propagator of the skeleton graphs the interacting two point function

$\frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)}$, where $\Sigma(k)$ is the proper self–energy. The proper self–energy is itself given by the sum of all amputated two–point skeleton diagrams with propagators $\frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)}$, so this prescription is implicit.

In practice, implementation of this resummation algorithm is not completely trivial, because it is not easy to verify that

$$\Sigma = \sum_{\substack{\text{all two–legged} \\ \text{1PI skeleton} \\ \text{diagrams } G}} \text{value of } G, \text{ using propagator } \frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(\mathbf{k})}$$

can be solved for Σ. In fact, it is far from obvious that the right hand side is even once differentiable with respect to Σ, because differentiating $\frac{1}{ik_0 - e(\mathbf{k}) - \Sigma(k)}$ once with respect to Σ produces a string of length two. And you will certainly not be able to solve for $\Sigma(k, \lambda) = \sum_{r=1}^{\infty} \lambda^r \Sigma_r(k)$ as a formal power series in λ, because the right hand side is certainly not C^∞ in Σ. However, there is a procedure that implements, at least the important part of, the above resummation algorithm and that can be mostly implemented in terms of formal power series.

This procedure, for generating well–defined terms in the perturbation expansion, effectively reparametrizes the family of models under consideration. Fix any model $(E(\mathbf{k}), \lambda)$. Suppose, for the time being, that you know the proper self–energy $\Sigma(k, E, \lambda)$ for this model. Write $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k})$ where $e(\mathbf{k})$ has the property that $\big\{ \mathbf{k} \bigm| e(\mathbf{k}) = 0 \big\}$ coincides with the interacting Fermi surface

$$F = \Big\{ \mathbf{k} \Bigm| E(\mathbf{k}) + \Sigma\big((0, \mathbf{k}), E, \lambda\big) = 0 \Big\}$$

The condition $\big\{ \mathbf{k} \bigm| e(\mathbf{k}) = 0 \big\} = F$ does not uniquely determine the decomposition $E = e + \delta e$. It only forces $\delta e(\mathbf{k}) = -\Sigma\big((0, \mathbf{k}), E, \lambda\big)$ for $\mathbf{k} \in F$. One can select a decomposition by specifying a "projection" P which maps each $\mathbf{k} \in \mathcal{D}$ to a unique $P\mathbf{k} \in F$. Then the decomposition is uniquely determined by the supplementary condition $\delta e(\mathbf{k}) = \delta e(P\mathbf{k})$.

If we formally expand

$$\begin{aligned} \frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)} &= \frac{1}{ik_0 - e(\mathbf{k}) - \delta e(\mathbf{k}) - \Sigma(k)} \\ &= \sum_{n=0}^{\infty} \frac{1}{ik_0 - e(\mathbf{k})} \left(\frac{\delta e(\mathbf{k}) + \Sigma(k)}{ik_0 - e(\mathbf{k})} \right)^n \end{aligned} \tag{I.3}$$

the numerator $\delta e(\mathbf{k})+\Sigma(k)$ vanishes on F, the zero set of the denominator, and the ratio $\frac{\delta e(\mathbf{k})+\Sigma(k)}{ik_0-e(\mathbf{k})}$ is locally L^∞ (assuming sufficient regularity and that the denominator has a simple zero). Thus each term in the expansion (I.3) is locally L^1.

Of course, in practice, $\Sigma(k,E,\lambda)$ is not known ahead of time, so this procedure has to be reordered. First, fix $e(\mathbf{k})$. Then define $\delta e(\mathbf{k})=\delta e(\mathbf{k},e,\lambda)$ to obey

$$\delta e(\mathbf{k})+\Sigma\big((0,\mathbf{k}),e+\delta e,\lambda\big)=0 \text{ for all } \mathbf{k} \text{ with } e(\mathbf{k})=0$$

This can be done by defining a projection P onto the interacting Fermi surface $F=\left\{\ \mathbf{k}\ \middle|\ e(\mathbf{k})=0\ \right\}$ and requiring

$$\delta e(\mathbf{k})=-\Sigma\big((0,P\mathbf{k}),e+\delta e,\lambda\big) \tag{I.4}$$

Observe that (I.4) is also an implicit equation for δe. However, the solubility of this equation in perturbation theory is trivial because Σ is $O(\lambda)$. Then define

$$E(\mathbf{k})=e(\mathbf{k})+\delta e(\mathbf{k},e,\lambda) \tag{I.5}$$

To end up with the $E(\mathbf{k})$ of (I.5) agreeing with the $E(\mathbf{k})$ we fixed a couple of paragraphs ago, we have to solve (I.5) for

$$e(\mathbf{k})=e(\mathbf{k},E,\lambda)$$

It looks like the invertibility of the map $e\mapsto E$ of (I.5) is again trivial in perturbation theory because $\delta e=O(\lambda)$. But, except in the rotationally invariant case (so that δe is independent of $\mathbf{k}$), it isn't because δe is not very regular. We do not treat the invertibility of (I.5) in this paper. It is treated in [FST1,2,3].

In this paper, we treat $e(\mathbf{k})$ as given and fixed and choose the counterterm $\delta e(\mathbf{k},\lambda)$ so that

$$\delta e(\mathbf{k},\lambda)=-\ell\Sigma\big(k,\lambda,\beta=\infty\big) \tag{I.6}$$

where the localization operator ℓ is a variant of

$$(\ell g)(k)=g\big((0,P\mathbf{k})\big)$$

that is smoothed off away from the Fermi surface. See §II.2 for details. Note that the counterterm is chosen independent of the temperature. We thus parametrize our family of models by (e,λ) rather than (E,λ). Consequently, $\frac{d}{d\lambda}$ means the derivative with respect to λ with e, rather than E, held fixed and the coefficient of λ^n in the Taylor expansion of the Euclidean Green's functions is the sum of all renormalized Feynman diagrams of order n. The Feynman rules for the renormalized diagrams consist of the Feynman rules of §I.2 supplemented by

- let $g(p)$ be the unrenormalized value of a two–legged subdiagram of G. Here p refers to the momentum flowing through the external lines of the subdiagram. Replace $g(p)$ by $g(p) - \ell g(p)$.

This renormalization operation is performed inductively from smaller to larger subgraphs.

§I.4 Main results

Consider the Euclidean Green's functions

$$\left\langle \prod_{i=1}^{N} \psi_{p_i,\sigma_i} \bar{\psi}_{q_i,\tau_i} \right\rangle_\beta = \frac{\int \left(\prod_{i=1}^N \psi_{p_i,\sigma_i}\bar{\psi}_{q_i,\tau_i}\right) e^{\mathcal{A}(\psi,\bar{\psi})} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar{\psi}_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar{\psi})} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar{\psi}_{k,\sigma}} \tag{I.7a}$$

for a model for which the free part of the action $\mathcal{A}$ is chosen to yield a propagator

$$\delta_{\sigma,\sigma'} \frac{\rho(|\mathbf{k}|/\mathfrak{C})}{ik_0 - e(\mathbf{k})} \tag{I.7b}$$

and the interaction part of the action is now given by

$$\begin{aligned} \mathcal{V}(\psi,\bar{\psi}) \;=\; & -\int đk\; \delta e(\mathbf{k},\lambda) \bar{\psi}_{k,\sigma}\psi_{k,\sigma} \\ & + \tfrac{\lambda}{2}\int \prod_{i=1}^{4} đk_i\; D\left(k_1{+}k_2{-}k_3{-}k_4\right)\; \bar{\psi}_{k_1,\sigma}\psi_{k_3,\sigma}\, \langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\, \bar{\psi}_{k_2,\tau}\psi_{k_4,\tau} \end{aligned} \tag{I.7c}$$

Here ρ is a C_0^∞ function that is one in a neighbourhood of zero and $\mathfrak{C}$ is fixed but arbitrary. This is how we introduce the ultraviolet cutoff. Because the argument of ρ does not involve k_0, this uv cutoff can be implemented by putting a uv cutoff in the Hamiltonian. The counterterm $\delta e(\mathbf{k},\lambda)$ is given using the localization operator of §II.2. We assume

H1) $e(\mathbf{k})$ is C^1

H2) $\nabla e(\mathbf{k}) \neq 0$ for all k with $e(\mathbf{k}) = 0$

H3) $\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle$ is C^1

H4) Let $\langle (t_1,\mathbf{k}_1),(t_2,\mathbf{k}_2)|\mathrm{V}|(t_3,\mathbf{k}_3),(t_4,\mathbf{k}_4)\rangle$ be the mixed time/spatial momentum space representation of $\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle$. That is, the Fourier transform in the temporal variables of $\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle$. Then

$$\max_{1\le j\le 4} \sup_{\substack{\mathbf{k}_1,\cdots,\mathbf{k}_4\\ t_j}} \int \Big| \prod_{i=1}^{4} \partial_{\mathbf{k}_i}^{\alpha_i} \prod_{i=1}^{4} [1+|t_i - t_j|]^{N_i} \, \langle (t_1,\mathbf{k}_1),(t_2,\mathbf{k}_2)|\mathrm{V}|(t_3,\mathbf{k}_3),(t_4,\mathbf{k}_4)\rangle \Big| \prod_{\substack{i=1\\ i\neq j}}^{4} dt_i < \infty$$

for all $|\alpha| \le 1$ and $\sum_i N_i \le 3$.

For any function $G(p_1, \sigma_1, \cdots, q_N, \tau_N)$, define the norm

$$\big|G\big| = \prod_{j=1}^{N} \Bigg\{ \sum_{\sigma_j,\tau_j} \int đp_j \, đq_j \; \frac{1}{[1+p_{0,j}^2][1+q_{0,j}]^2} \Bigg\} |G(p_1, \sigma_1, \cdots, q_N, \tau_N)|$$

Our main result is

Theorem I.2 *Assume Hypotheses H1,2,3,4. Let $G(\vec{p}, \beta)$ be the amplitude of any graph contributing to the $2N$–point connected Euclidean Green's functions of the model (I.7). Let $\Sigma(p, \beta)$ be any graph contributing to the proper self–energy of the model (I.7). Then, for every $0 \le \epsilon < 1$,*

$$\sup_{\beta} \big|G(\,\cdot\,, \beta)\big| < \infty$$

$$\sup_{\beta} \beta^{\epsilon} \big|G(\,\cdot\,, \beta) - G(\,\cdot\,, \infty)\big| < \infty$$

$$\sup_{p,\beta} |\Sigma(p, \beta)| < \infty$$

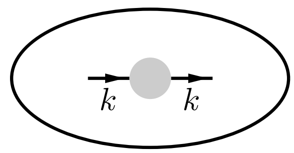
$$\sup_{p,\beta} \beta^{\epsilon} |\Sigma(p, \beta) - \Sigma(p, \infty)| < \infty$$

Theorem I.2 is an amalgam of Corollary II.5, Proposition II.6 and Proposition III.1. In a companion paper [FKST1] we prove similar, pointwise, bounds on the Bethe–Salpeter kernel.

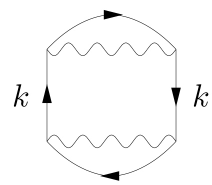
§I.5 Anomalous graphs

Kohn and Luttinger introduced, in [KL], a class of Feynman diagrams that they called "anomalous diagrams". There have been subsequent text book discussions in [FW, NO].

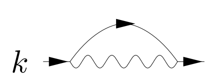
The [KL] definition was “anomalous diagrams are those for which momentum conservation forces some hole and electron lines to represent the same state.” Anomolous diagrams contain at least one two–legged subdiagram



One such diagram that they considered explicitly was

$\Omega_{2A} =$ 

Kohn and Luttinger concluded that such diagrams vanish if one holds the spatial volume fixed and finite and takes the limit as the temperature T tends to zero but do not vanish, for systems without spherical symmetry, if one takes the infinite volume limit before taking the temperature zero limit.

We claim that their conclusion was the consequence of an excessively restrictive renormalization prescription. In fact, their prescription is not sufficient to yield well–defined temperature zero limits of higher order vacuum diagrams in systems without spherical symmetry. We first review the argument of [KL], using Ω_{2A} as an illustration. Denote by $\Sigma_{2A}(\mathbf{k})$ the unrenormalized value of  . We are assuming that the interaction $\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle$ is given by a two–body potential $u(\mathbf{k}_1 - \mathbf{k}_3)$ so that Σ_{2A} is independent of k_0. First suppose that no renormalization is done. Then, at positive temperature and in a finite volume,

$$\begin{aligned}\Omega_{2A}(\beta, L) &= 2\frac{1}{\beta}\sum_{k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\frac{1}{L^d}\sum_{\mathbf{k}\in\frac{2\pi}{L}\mathbb{Z}}\frac{\Sigma_{2A}(\mathbf{k})^2}{[ik_0 - e(\mathbf{k})]^2}\\ &= -2\frac{1}{L^d}\sum_{\mathbf{k}\in\frac{2\pi}{L}\mathbb{Z}}\Delta\big(e(\mathbf{k})\big)\Sigma_{2A}(\mathbf{k})^2 \qquad \text{(I.8) = [KL eqn. (22)]}\end{aligned}$$

where

$$\Delta(\epsilon) = \beta\frac{e^{\beta\epsilon}}{[e^{\beta\epsilon}+1]^2}$$

In the limit as $\beta \to \infty$, $\Delta(\epsilon)$ becomes the Dirac delta function $\delta(\epsilon)$. If we hold L fixed, such that $e(\mathbf{k})$ never vanishes for $\mathbf{k} \in \frac{2\pi}{L}\mathbb{Z}$, then $\lim\limits_{\beta\to\infty} \Omega_{2A}(\beta, L) = 0$. Taking the infinite volume limit (requiring L to always obey the above irrationality condition),

$$\lim_{L\to\infty} \lim_{\beta\to\infty} \Omega_{2A}(\beta, L) = 0$$

On the other hand

$$\begin{aligned} \lim_{\beta\to\infty} \lim_{L\to\infty} \Omega_{2A}(\beta, L) &= \lim_{\beta\to\infty} -2 \int \tfrac{d^d\mathbf{k}}{(2\pi)^d}\ \Delta\big(e(\mathbf{k})\big)\Sigma_{2A}(\mathbf{k})^2 \\ &= -2 \int \tfrac{d^d\mathbf{k}}{(2\pi)^d}\ \delta\big(e(\mathbf{k})\big)\Sigma_{2A}(\mathbf{k})^2 \end{aligned} \tag{I.9}$$

which need not vanish if $\Sigma_{2A}(\mathbf{k})$ does not vanish on the Fermi surface $e(\mathbf{k}) = 0$. This is generically so, even in the spherical case. For example, at temperature zero, when the two–body potential u is a delta function, $\Sigma_{2A}(\mathbf{k})$ is just $\frac{1}{(2\pi)^d}$ times the volume contained by the Fermi surface.

Upon renormalization, $\Sigma_{2A}(\mathbf{k})$ is replaced by $\Sigma_{2A}(\mathbf{k}) - (\ell\Sigma_{2A})(\mathbf{k})$ where ℓ is the localization operator determined by the renormalization prescription. Then (I.9) is replaced by

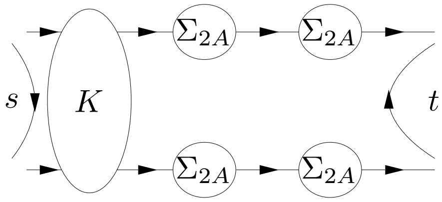
$$\lim_{\beta\to\infty} \lim_{L\to\infty} \Omega_{2A}(\beta, L) = -2 \int \tfrac{d^d\mathbf{k}}{(2\pi)^d}\ \delta\big(e(\mathbf{k})\big)\big[\Sigma_{2A}(\mathbf{k}) - (\ell\Sigma_{2A})(\mathbf{k})\big]^2 \tag{I.10}$$

Equation (17) of [KL] implements renormalization of the chemical potential. When this is the only renormalization, $(\ell\Sigma_{2A})(\mathbf{k})$ must necessarily be a constant, independent of $\mathbf{k}$. So, unless $\Sigma_{2A}(\mathbf{k})$ is also constant on the Fermi surface, (I.10) will, once again, fail to vanish. In the spherically symmetric case, $\Sigma_{2A}(\mathbf{k})$ is rotationally invariant and hence constant on the Fermi surface so that (I.10) vanishes. However, generically, when spherical symmetry is broken, $\Sigma_{2A}(\mathbf{k})$ is not constant on the Fermi surface and (I.10) is nonzero. This is precisely the conclusion of [KL].

Using a renormalization prescription that does not force $\Sigma_{2A}(\mathbf{k}) - (\ell\Sigma_{2A})(\mathbf{k})$, and similar expressions for other two–legged subdiagrams, to vanish on the Fermi surface at temperature zero can have even more dire consequences than noncommutation of the limits $\lim_{\beta\to\infty}$ and $\lim_{L\to\infty}$. Consider for example the Bethe–Salpeter equation

$$\chi(s) = -\tfrac{1}{\beta} \sum_{t_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \tfrac{d^d\mathbf{t}}{(2\pi)^d} K(s,t)G(t)G(-t)\chi(t) \tag{I.11}$$

Here $K(s,t)$ is the Bethe–Salpeter kernel, at zero transfer momentum, for a spin independent interaction, and $G(t)$ is the interacting two–point function. Diagrams of the form



contribute, including the effects of renormalization,

$$-\tfrac{1}{\beta}\sum_{t_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\int \tfrac{d^d\mathbf{t}}{(2\pi)^d}K(s,t)\tfrac{|\Sigma_{2A}(t)-(\ell\Sigma_{2A})(t)|^{2n}}{|it_0-e(\mathbf{t})|^{2(n+1)}}\chi(t)$$

to the right hand side of (I.11). Here n is the number of Σ_{2A}'s on each string. As n grows, the singularity of $\frac{1}{|it_0-e(\mathbf{t})|^{2(n+1)}}$ at $t_0=0,\ e(\mathbf{t})=0$ becomes more and more severe, unless there is a compensating vanishing of $\Sigma_{2A}(t)-(\ell\Sigma_{2A})(t)$. For large β and $n>0$

$$\tfrac{1}{\beta}\sum_{t_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\int \tfrac{d^d\mathbf{t}}{(2\pi)^d}\tfrac{1}{|it_0-e(\mathbf{t})|^{2(n+1)}}\sim \text{const}\,\beta^{2n}$$

rather than the usual $\ln\beta$. Including such terms in (I.11) gives entirely wrong behaviour for the critical temperature.

The cure for both the problem of the last paragraph and the anomalous diagram problem is to use a renormalization prescription which ensures that $\Sigma_{2A}(\mathbf{k})-(\ell\Sigma_{2A})(\mathbf{k})$, and similar expressions for other two–legged subdiagrams, vanish for $k_0=0,\ e(\mathbf{k})=0$. Then the right hand side of (I.10) is zero, because $\big[\Sigma_{2A}(\mathbf{k})-(\ell\Sigma_{2A})(\mathbf{k})\big]^2$ is zero on the support of the delta function $\delta\big(e(\mathbf{k})\big)$. And, for all n, $\frac{|\Sigma_{2A}(t)-(\ell\Sigma_{2A})(t)|^{2n}}{|it_0-e(\mathbf{t})|^{2(n+1)}}$ has the same degree of singularity as $\frac{1}{|it_0-e(\mathbf{t})|^{2}}$. One such renormalization prescription is $(\ell g)(k)=g\big((0,P\mathbf{k})\big)$, where P is a projection onto the Fermi surface. Another is defined in §II.2. Yet another is the skeleton diagram prescription under which the localization operator is set to the identity, so that the full interacting two–point function $G(k)=\frac{1}{ik_0-E(\mathbf{k})-\Sigma(k)}$ is used as the propagator and no nontrivial two-legged subdiagram ever appears.

§II. The Infrared End

This chapter is the heart of this paper. We prove bounds uniform in β on, and convergence as $\beta \to \infty$ of, graphs arising from a wide class of models. In all of these models, the propagator is

$$C(k) = \delta_{\sigma,\sigma'} \frac{\rho\big(k_0^2 + e(\mathbf{k})^2\big)}{ik_0 - e(\mathbf{k})}$$

where ρ is a C_0^∞ function that is one in a neighbourhood of zero and

H1) $e(\mathbf{k})$ is C^1 and $\rho\big(k_0^2 + e(\mathbf{k})^2\big)$ has compact support in $\mathbb{R}^{d+1}$

H2) $\nabla e(\mathbf{k}) \neq 0$ for all k with $e(\mathbf{k}) = 0$

H3ir) Every interaction vertex has an even number of external (particle) legs. If the vertex has two legs, its vertex function $u_v(k)$ vanishes when $k_0 = 0$ and $\mathbf{k} \in F$.

H4ir) The kernel of each interaction vertex is a C^1 function of the momenta flowing into the vertex. It is also a C^1 function of $1/\beta$ at $\beta = \infty$.

Hypothesis (H1) requires that all momenta $k = (k_0, \mathbf{k})$ run over a compact set. However, because we do not require interaction vertices to have four legs, the interaction may be the effective interaction obtained by "integrating out" the ultraviolet end of a model in the class specified in §I. We discuss the ultraviolet end of the model in §III.

§II.1 Propagator Bounds

To analyse graphs we express the propagator as a sum

$$C(k) = \sum_{j=-\infty}^{0} C^{(j)}(k)$$

of "scale" propagators. Roughly speaking, the propagator of scale j is the part of $C(k)$ that has magnitude M^{-j}, where $M > 1$ is just the constant that sets the scale units. We define

$$C^{(j)}(k) = \delta_{\sigma,\sigma'} \frac{f\big(M^{-2j}[k_0^2 + e(\mathbf{k})^2]\big)}{ik_0 - e(\mathbf{k})}$$

with f being a C^∞ function with support in $[M^{-4}, 1]$ that obeys

$$\sum_{j=-\infty}^{0} f\big(M^{-2j}x\big) = \rho(x) \qquad \text{for all } x > 0$$

It is easy [FT1] to choose f and ρ satisfying the specified conditions.

Also define $[k_0]_\beta$ to be the element of $\frac{\pi}{\beta}(2\mathbb{Z}+1)$ nearest k_0. Use any tie breaking rule you like. Note that $C^{(j)}$ is defined and C^1 for all k, including all k_0. The properties of $C^{(j)}$ that we use are given in the following Lemma.

Lemma II.1

a)

$$\sup_{k\in\mathbb{R}^{d+1}} \big|C^{(j)}(k)\big| \le M^2 M^{-j}$$

b) If $e(\mathbf{k})$ is C^r and $|\alpha|\le r$, then

$$\sup_{k\in\mathbb{R}^{d+1}} \big|\partial_k^\alpha C^{(j)}(k)\big| \le \mathrm{const}_{|\alpha|}\, M^{-(1+|\alpha|)j}$$

c) If $k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)$ and $C^{(j)}(k)\ne 0$ then $M^j\ge\frac{\pi}{\beta}$. That is $j\ge-\log_M(\beta/\pi)$.

d) For every $0\le\epsilon\le 1$ and $n\ge 1$

$$\left|C^{(j)}\Big(\sum_{i=1}^n k_i\Big) - C^{(j)}\Big(\sum_{i=1}^n([k_{i,0}]_\beta,\mathbf{k}_i)\Big)\right| \le \mathrm{const}\,\frac{n^\epsilon}{\beta^\epsilon}M^{-(1+\epsilon)j}$$

e)

$$\tfrac{1}{\beta}\#\Big\{\ k_0\in\tfrac{\pi}{\beta}(2\mathbb{Z}+1)\ \Big|\ C^{(j)}(k)\ne 0 \qquad \text{for some } \mathbf{k}\ \Big\} \le \tfrac{2}{\pi}M^j$$

f)

$$\mathrm{Vol}\Big\{\ \mathbf{k}\in\mathbb{R}^d\ \Big|\ C^{(j)}(k)\ne 0 \qquad \text{for some } k_0\in\mathbb{R}\ \Big\} \le \mathrm{const}\, M^j$$

Proof: a, b) In order for k to be in the support of $C^{(j)}$ it is necessary that

$$M^{-2j}\big|k_0^2+e(\mathbf{k})^2\big| \ge M^{-4}$$

or equivalently

$$\big|ik_0+e(\mathbf{k})\big| \ge M^{-2}M^j$$

By Leibnitz, acting on $C^{(j)}$ by a multiple derivative ∂^α gives a finite linear combination of terms of the form

$$\frac{\partial^\beta f\big(M^{-2j}[k_0^2+e(\mathbf{k})^2]\big)}{ik_0-e(\mathbf{k})}\prod_i \frac{\partial^{\gamma_i}\left(ik_0-e(\mathbf{k})\right)}{ik_0-e(\mathbf{k})}$$

with $\beta+\sum_i \gamma_i=\alpha$. As, on the support of $C^{(j)}$,

$$M^{-2}M^j \le \big|ik_0+e(\mathbf{k})\big| \le M^j$$

the desired result follows from

$$\sup_x |f^{(n)}(x)| \le \mathrm{const}_n$$
$$\sup_k |\partial^\gamma e(\mathbf{k})| \le \mathrm{const}_\gamma$$

c) To have $C^{(j)}(k)\ne 0$, it is necessary that $\big|ik_0+e(\mathbf{k})\big|\le M^j$ and hence that $\big|k_0\big|\le M^j$. As the smallest element of $\frac{\pi}{\beta}(2\mathbb{Z}+1)$ is of modulus $\frac{\pi}{\beta}$, this forces $\frac{\pi}{\beta}\le M^j$.

d) By definition

$$|k_0-[k_0]_\beta| \le \frac{\pi}{\beta}$$

so, $|\sum_{i=1}^n k_{i,0}-\sum_{i=1}^n [k_{i,0}]_\beta| \le \dfrac{n\pi}{\beta}$ and, by the mean value theorem,

$$\Big|C^{(j)}\Big(\sum_{i=1}^n k_i\Big)-C^{(j)}\Big(\sum_{i=1}^n([k_{i,0}]_\beta,\mathbf{k}_i)\Big)\Big| \le \frac{n\pi}{\beta}\sup_p\Big|\partial_{p_0}C^{(j)}(p)\Big| \le \mathrm{const}\,\frac{n}{\beta}M^{-2j}$$

Taking a weighted geometric mean of this with

$$\Big|C^{(j)}\Big(\sum_{i=1}^n k_i\Big)-C^{(j)}\Big(\sum_{i=1}^n([k_{i,0}]_\beta,\mathbf{k}_i)\Big)\Big| \le 2\sup_p\Big|C^{(j)}(p)\Big| \le \mathrm{const}\,M^{-j}$$

gives the desired bound.

e,f) In order for $C^{(j)}(k)$ to be nonzero it is necessary that

$$\big|ik_0+e(\mathbf{k})\big| \le M^j$$

and hence that

$$|k_0|\le M^j \qquad \text{and} \qquad |e(\mathbf{k})|\le M^j$$

The separation between neighbouring values of k_0 is $\frac{2\pi}{\beta}$. So the maximum number of allowed k_0's is the length of the interval from $-M^j-\frac{\pi}{\beta}$ to $M^j+\frac{\pi}{\beta}$ divided by $\frac{2\pi}{\beta}$. In other words $\frac{\beta M^j}{\pi}+1$, which is no more than $\frac{2M^j}{\pi}\beta$, by part c). The bound on the volume in $\mathbf{k}$ space follows from the requirement that $\nabla e(\mathbf{k})$ be bounded away from zero and continuous and that the Fermi surface be compact. ∎

Corollary II.2 *There is a constant independent of $\beta > 1$ such that*

$$\begin{aligned} \left\| C^{(j)} \right\|_\infty &\equiv \sup_{\substack{k_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1) \\ \mathbf{k} \in \mathbb{R}^d}} \left| C^{(j)}(k) \right| && \le M^2 M^{-j} \\ \left\| C^{(j)} \right\|_1 &\equiv \tfrac{1}{\beta} \sum_{k_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} \int \bar{d}^d\mathbf{k} \ \left| C^{(j)}(k) \right| && \le \text{const}\, M^j \end{aligned}$$

§II.2 Localization

We now give a precise definition of the localization operator used in this paper. First, suppose that $e(\mathbf{k})$ and the kernels of all interaction vertices are rotation invariant. Then, for any quadratic element of the Grassmann algebra of the form $\int \bar\psi(k) H(k,\beta)\psi(k)$ set

$$L \int dk\ \bar\psi(k) H(k,\beta)\psi(k) = \int dk\ \bar\psi(k)(\ell H)(k)\psi(k)$$

where $P\mathbf{k} = \frac{k_F}{|\mathbf{k}|}\mathbf{k}$, $k' = (0, P\mathbf{k})$ is the projection onto the Fermi surface and

$$(\ell H)(k) = H(k', \infty)$$

In this case, the kernel of a graph contributing to the proper self energy is of the form $H(k,\beta) = \tilde H\big(k_0, e(\mathbf{k}), \beta\big)$ so that

$$(\ell H)(k) = \tilde H(0,0,\infty)$$

It is easy to control this localization operator because ℓH is independent of β and k. In particular $\partial_k \ell H = 0$. We have

Lemma II.3 *Let k lie in the support of $C^{(j)}(k)$ with $M^j \ge \frac{\pi}{\beta}$. Then,*

(a) $$|(\ell H)(k)| \le \sup_p |H(p,\infty)|$$

(b) $$|\partial_k (\ell H)(k)| \le \text{const}\, M^{-j} \sup_p |H(p,\infty)|$$

(c) $$|H(k,\beta) - (\ell H)(k)| \le \text{const}\, M^j \sup_p |\partial_p H(p,\infty)| + \sup_p |H(p,\beta) - H(p,\infty)|$$

(d) $$\begin{aligned} \big|\partial_k \big(H(k,\beta) - (\ell H)(k)\big)\big| \le \text{const} & \max_{\beta' \in \{\beta,\infty\}} \sup_p |\partial_p H(p,\beta')| \\ & + \text{const}\, M^{-j} \sup_p |H(p,\beta) - H(p,\infty)| \end{aligned}$$

For $\beta < \infty$, the derivative ∂_{k_0} is defined by

$$\partial_{k_0} f(k_0) = \tfrac{\beta}{2\pi}\big[f\big(k_0 + \tfrac{2\pi}{\beta}\big) - f(k_0)\big]$$

The construction of an appropriate localization operator for a general, non rotation invariant, model is more involved. If we were to retain the definition $(\ell H)(k) = H\big((0, P\mathbf{k}), \infty\big)$, with $P\mathbf{k}$ being some suitable projection on the Fermi surface, then, not only would ℓH no longer be a constant, but Lemma II.3 would fail. The ∂_k could have a nontrivial action on $H((0, P\mathbf{k}), \infty)$. This is bad because the scale of some lines of H can be much lower than that of the momentum k entering H. Fortunately, it is possible to extend the definition of ℓ to the non-rotationally invariant case so that Lemma II.3 remains valid.

To do this, we first construct a partition of unity for a neighbourhood of the Fermi surface F. Let $\mathcal{N}$ be a neighbourhood of F that is diffeomorphic to $(-1,1) \times F$. The partition of unity is of the form

$$\mathbf{k} \in \mathcal{N}, e(\mathbf{k}) \neq 0 \Longrightarrow \sum_{j \le 0} \sum_{\Sigma \in S_j} f_j(k) \chi_\Sigma(\mathbf{k}) = 1$$

where

$$f_j(k) = f\big(M^{-2j}[k_0^2 + e(\mathbf{k})^2]\big)$$

is our standard scale j cutoff that restricts $\sqrt{k_0^2 + e(\mathbf{k})^2}$ to lie between $M^{-2}M^j$ and M^j and $\chi_\Sigma(\mathbf{k})$ depends only on the angular components of $\mathbf{k}$ (that is, all but the first coordinate of the diffeomorphism) and restricts those components to lie in a patch on F of diameter M^j. Hence S_j is a list of about const $M^{-(d-1)j}$ patches on F. Furthermore, for each $\Sigma \in S_j$ there is a point $\mathbf{k}_{j,\Sigma} \in F$ that is at most distance const M^j from every k in the support of $f_j(k)\chi_\Sigma(\mathbf{k})$. We define

$$(\ell H)(k) = \sum_{j \le 0} \sum_{\Sigma \in S_j} f_j(k) \chi_\Sigma(\mathbf{k}) H\big(0, \mathbf{k}_{j,\Sigma}, \infty\big)$$

If $k_0 = 0$ and $\mathbf{k}$ is exactly on the Fermi surface, every $f_j(k) = 0$ and we define $\ell H(k)$ by taking limits in the formula above as $p \to k$ with $p_0 \neq 0$ or $\mathbf{p}$ not on the Fermi surface. This gives

$$(\ell H)(k) = H(k, \infty)$$

With this definition, we can give the

Proof of Lemma II.3 for a general model:

(a) is obvious because $\sum_{j\le 0}\sum_{\Sigma\in S_j} f_j(k)\chi_\Sigma(\mathbf{k})=1$.

(b) First observe that for any fixed k in the support of $C^{(j)}$, there are at most const, independent of j,k, pairs (j',Σ) for which $f_{j'}(k)\chi_\Sigma(\mathbf{k})\ne 0$ and that $|j'-j|\le 1$ for all of these pairs. Hence, it suffices to prove the bounds for each of the pairs. Next observe that the $H\big((0,\mathbf{k}_{j',\Sigma}),\infty\big)$ are independent of k and bounded by $\sup_p |H(p,\infty)|$. Finally, observe that, as usual, any derivative acting on $f_{j'}(k)\chi_\Sigma(\mathbf{k})$ costs $\mathrm{const}\, M^{-j'}$. This also applies to the discrete ∂_{k_0} derivative by the Mean Value Theorem.

(c, d) Write

$$H(k,\beta)-(\ell H)(k)=\sum_{j\le 0}\sum_{\Sigma\in S_j} f_j(k)\chi_\Sigma(\mathbf{k})\big[H(k,\beta)-H\big(0,\mathbf{k}_{j,\Sigma},\infty\big)\big]$$

As in (b), for any fixed k in the support of $C^{(j)}$, there are at most const, independent of j,k, pairs (j',Σ) for which $f_{j'}(k)\chi_\Sigma(\mathbf{k})\ne 0$. Furthermore $|k_0|\le \mathrm{const}\, M^j$ and, as $|j'-j|\le 1$ for all of these pairs, $|\mathbf{k}-\mathbf{k}_{j',\Sigma}|\le \mathrm{const}\, M^j$ for all of these pairs. So part (c) follows from the Mean Value Theorem. If the derivative of (d) acts on the partition of unity, we again apply the Mean Value Theorem. ∎

§II.3 Bounds on the proper self-energy

Let G be a Feynman diagram. As is the case throughout this chapter, G has only particle lines and every interaction vertex of G has an even number (not necessarily four) of external legs. We also use the symbol G to stand for the value of the graph G. First, suppose that G is not renormalized. Expand each propagator of G using $C=\sum_{j\le 0} C^{(j)}$ to give

$$G=\sum_J G^J$$

The sum runs over all possible labellings of the graph G, with each labelling consisting of an assignment $J=\big\{\, j_\ell\le 0 \,\big|\, \ell\in G\,\big\}$ of scales to the lines of G. We now construct a natural hierarchy of subgraphs of G^J. This family of subgraphs will be a forest, meaning that if

$G_f, G_{f'}$ are in the forest and intersect, either $G_f \subset G_{f'}$ or $G_{f'} \subset G_f$. First let, for each $j \le 0$,

$$G^{(\ge j)} = \left\{ \ \ell \in G^J \ \middle| \ j_\ell \ge j \ \right\}$$

be the subgraph of G^J consisting of all lines of scale at least j. There is no need for $G^{(\ge j)}$ to be connected. The forest $t(G^J)$ is the set of all connected subgraphs of G^J that are components of some $G^{(\ge j)}$. This forest is naturally partially ordered by containment. In order to make $t(G^J)$ look like a tree with its root at the bottom, we define, for $f, f' \in t(G^J)$, $f > f'$ if $G_f \subset G_{f'}$. We denote by $\pi(f)$ the highest fork of $t(G^J)$ below f and by ϕ the root element, i.e. the element with $G_\phi = G$. To each $G_f \in t(G)$ there is naturally associated the scale $j_f = \min\left\{ \ j_\ell \ \middle| \ \ell \in G_f \ \right\}$.

Reorganize the sum over J using

$$G = \sum_{t \in \mathcal{F}(G)} \sum_{j \le 0} \sum_{J \in \mathcal{J}(j,t,G)} G^J \tag{II.1}$$

where

$$\mathcal{F}(G) = \text{ the set of forests of subgraphs of } G$$

$$\mathcal{J}(j,t,G) = \left\{ \text{ labellings } J \text{ of } G \ \middle| \ t(G^J) = t, \ j_\phi = j \ \right\}$$

A given labelling J of G is in $\mathcal{J}(j,t,G)$ if and only if

- for each $f \in t$, all lines of $G_f \setminus \cup_{\substack{f' \in t \\ f' > f}} G_{f'}$ have the same scale. Call the common scale j_f.
- if $f > f'$ then $j_f > j_{f'}$
- $j_\phi = j$

It is a standard result ([G,GN,F],[FT1§6], [FT2§1]) that the renormalization prescription discussed in §I.3 may be implemented by modifying II.1 as follows.

- each $f \in t$ for which G_f has two external lines is assigned a "renormalization label". This label can take the values r and c. The set of possible assignments of renormalization labels, i.e. the set of all maps from $\left\{ \ f \in t \ \middle| \ G_f \text{ has two external legs} \ \right\}$ to $\{r, c\}$, is denoted $\mathcal{R}(t)$.
- in the definition of the renormalized value of the graph G, the value of each subgraph G_f with renormalization label r is replaced by $(\mathbb{1} - \ell) G_f(k)$. Here ℓ is the localization operator defined in §II.2. For these r–forks, the constraint $j_f > j_{\pi(f)}$ still applies.

Two–legged vertices v of G may, by the hypothesis in (H3ir) that $\ell u_v = 0$, be treated as r–forks, though, in this case, j_f is held fixed at 0.

- in the definition of the renormalized value of the graph G, the value of each subgraph G_f with renormalization label c is replaced by $\ell G_f(k)$. For these c forks the constraint $j_f > j_{\pi(f)}$ is replaced by $j_f \le j_{\pi(f)}$.

Given a graph G, a forest t of subgraphs of G and an assignment R of renormalization labels to the two–legged forks of t, we define $\mathcal{J}(j,t,R,G)$ to be the set of all assignments of scales to the lines of G obeying

- for each $f \in t$, all lines of $G_f \setminus \cup_{\substack{f' \in t \\ f' > f}} G_{f'}$ have the same scale. Call the common scale j_f.
- if G_f is not two–legged then $j_f > j_{\pi(f)}$
- if G_f is two–legged and $R_f = r$ and G_f is not a single two–legged vertex, then $j_f > j_{\pi(f)}$
- if G_f is two–legged and $R_f = r$ and G_f is a single two–legged vertex, then $j_f = 0$
- if G_f is two–legged and $R_f = c$ then $j_f \le j_{\pi(f)}$
- $j_\phi = j$

Then, the value of the graph G with all two–legged subdiagrams correctly renormalized is

$$G = \sum_{t \in \mathcal{F}(G)} \sum_{R \in \mathcal{R}(t)} \sum_{j \le 0} \sum_{J \in \mathcal{J}(j,t,R,G)} G^J \tag{II.1r}$$

Note that $\mathcal{F}(G)$ and $\mathcal{R}(t)$ are both finite sets with cardinalities that depend on the graph G but not on the temperature β. So to prove that G is well–defined and continuous in β it suffices to prove that, for each fixed t and R, $\sum_{j \le 0} \sum_{J \in \mathcal{J}(j,t,R,G)} G^J$ is well–defined and continuous in β. To derive bounds on G, when we are not interested in the dependence of those bounds on G and in particular on the order of perturbation theory, it suffices to derive bounds on $\sum_{j \le 0} \sum_{J \in \mathcal{J}(j,t,R,G)} G^J$ for each fixed t and R.

The basic bound on the proper self energy is

Proposition II.4 *Let G be a two–legged graph with n vertices and L internal lines. Let t be a tree corresponding to a forest of subgraphs of G. Let R be an assignment of r,c labels to all forks $f > \phi$ of t for which G_f is two–legged. Let $\mathcal{J}(j,t,R,G)$ be the set of all assignments of* *scales to the lines of G that have root scale j and are consistent with t and R. Then there is a constant* $\mathrm{const}_{n,L}$ *(depending on n, L, M and the vertex functions u_v but independent of β, G, t and j), such that for $|s| \in \{0,1\}$*

$$\sum_{J \in \mathcal{J}(j,t,R,G)} \sup_{p,\beta} \left|\partial_p^s G^J(p,\beta)\right| \le \mathrm{const}_{n,L}\, |j|^{L-1} M^{j(1-s)}$$

$$\sum_{J \in \mathcal{J}(j,t,R,G)} \sup_{p,\beta} \beta \left|G^J(p,\beta) - G^J(p,\infty)\right| \le \mathrm{const}_{n,L}\, |j|^{L-1}$$

For $\beta < \infty$, the derivative ∂_{p_0} is defined by

$$\partial_{p_0} f(p_0) = \tfrac{\beta}{2\pi} \left[f\left(p_0 + \tfrac{2\pi}{\beta}\right) - f(p_0) \right]$$

Remark. Note that here the root scale is not summed over and G_ϕ is not renormalized. But all internal scales are summed over and internal two–legged subgraphs that correspond to r and c forks are renormalized and localized respectively.

Remark. These bounds are fairly crude. The sum over root scale j diverges for the first right hand side with $|s| = 1$ and for the second right hand side. We prove tighter bounds in [FKST1], for a more restricted but still wide class of models, that imply that G is C^1 in p. For still tighter bounds see [FST2].

Proof: Define

$$\partial_T f(p,\beta) = \beta[f(p,\beta) - f(p,\infty)]$$

We must bound

$$\sum_{J \in \mathcal{J}(j,t,R,G)} \sup_{p} \left|\partial_p^s \partial_T^{s'} G^J(p,\beta)\right|$$

for all s, s' with $|s| + s' = 0, 1$. The proof is by induction on the depth of the graph. The depth is defined by

$$D = \max\left\{\, n \;\middle|\; \exists \text{ forks } f_1 > f_2 > \cdots > f_n > \phi \text{ with } G_{f_1}, \cdots, G_{f_n} \text{ all two–legged } \right\}$$

Two–legged vertices, if any, of G are to be treated as r–forks with scale zero.

case $D = s = s' = 0$. In this case, no proper subgraph $G_f,\ f \in t$ is two–legged so there is no renormalization to worry about. The form of the integral defining $G^J(p,\beta)$ is

$$G^J(p,\beta) = \int \prod_{\ell\in G\setminus T} đk_\ell \prod_{\ell\in G} C^{(j_\ell)}(k_\ell) \prod_v u_v(\vec{k}_v,\beta)$$

Here T is any spanning tree for G. As discussed in §I.2, there is associated with each spanning tree a complete set of momentum loops. The loops are labelled by the lines of $G \setminus T$. For each $\ell \in T$, the momentum k_ℓ is a signed sum of loop momenta and external momentum p. The product $\prod_v$ runs over the vertices of G. The set of all momenta entering a vertex v has been denoted

$$\vec{k}_v = \left\{ \ k_\ell \ \middle| \ v \text{ is at one end of } \ell \ \right\}$$

We shall prove several bounds on integrals like that for $G^J(p,\beta)$ in this paper and its companion [FKST1]. A common strategy will be used to prove all of the bounds. We have numbered the main steps in the most involved proof 1 through 8. We will use the same numbering in all of the proofs. However, in the easier arguments, like the one we are about to start now, some of the steps are skipped.

(1) Choose a spanning tree T for G with the property that $T \cap G_f^J$ is a connected tree for every $f \in t(G^J)$. T can be built up inductively, starting with the smallest subgraphs G_f, because, by construction, every G_f is connected and $t(G^J)$ is a forest.

(2) will be the application of $\partial_p^s \partial_T^{s'}$. It is not used in this case.

(3) will be the bounding of two–legged renormalized subgraphs (i.e. r–forks) and two–legged counterterms (i.e. c–forks). It is not used in this case.

(4) Bound all vertex functions, u_v, by their suprema in momentum space. We now have

$$\|G^J\|_\infty \le \prod_v \|u_v\|_\infty \int \prod_{\ell\in G\setminus T} đk_\ell \prod_{\ell\in G} |C^{(j_\ell)}(k_\ell)|$$

(5) will be used in the extraction of volume improvement factors from overlapping loops. It is not used in this paper.

(6) Use Lemma II.1a) to bound the propagator of each line in T by its supremum. We now have

$$\|G^J\|_\infty \le \prod_{\ell\in T} M^{2-j_\ell} \prod_v \|u_v\|_\infty \int \prod_{\ell\in G\setminus T} đk_\ell \prod_{\ell\in G\setminus T} |C^{(j_\ell)}(k_\ell)|$$

(7) will also be used in the extraction of volume improvement factors from overlapping loops. It is not used in this paper.

(8) Integrate over the remaining loop momenta. Each remaining loop momentum now just appears in a single factor - the propagator of the unique line of the loop not in T. Integration over each remaining loop gives the L^1 norm of that propagator, which has been bounded in Corollary II.2.

The above eight steps give

$$\|G^J\|_\infty \le \mathrm{const}^n \prod_{v\in G} \|u_v\|_\infty \prod_{\ell\in T} M^{-j_\ell} \prod_{\ell\in G^J\setminus T} \mathrm{const}\, M^{j_\ell}$$

Define the notation

$$\begin{aligned} T_f &= \text{number of lines of } T\cap G_f \\ L_f &= \text{number of internal lines of } G_f \\ n_f &= \text{number of vertices of } G_f \\ E_f &= \text{number of external lines of } G_f \\ E_v &= \text{number of lines hooked to vertex } v \\ \pi(v) &= \max\big\{\ f\in t(G^J)\ \big|\ \text{the vertex } v \text{ is in } G_f\ \big\} \end{aligned}$$

Applying

$$M^{\alpha j_\ell} = M^{\alpha j_\phi} \prod_{\substack{f\in t\\ f>\phi \\ \ell\in G_f}} M^{\alpha(j_f - j_{\pi(f)})}$$

to each $M^{\pm j_\ell}$ and

$$1 \le M^{-\frac{1}{2}(E_v-4)j_{\pi(v)}} = M^{-\frac{1}{2}(E_v-4)j_\phi} \prod_{\substack{f\in t\\ f>\phi \\ v\in G_f}} M^{-\frac{1}{2}(E_v-4)(j_f-j_{\pi(f)})}$$

for each vertex v gives

$$\|G^J\|_\infty \le \mathrm{const}^{n+L}\, M^{j(L_\phi - 2T_\phi - \sum_{v\in G}\frac{1}{2}(E_v-4))} \prod_{\substack{f\in t\\ f>\phi}} M^{(j_f-j_{\pi(f)})(L_f-2T_f-\sum_{v\in G_f}\frac{1}{2}(E_v-4))}$$

As

$$\begin{aligned} L_f &= \tfrac{1}{2}\big(\sum_{v\in G_f} E_v - E_f\big) \\ T_f &= n_f - 1 = \sum_{v\in G_f} 1 - 1 \\ \Longrightarrow L_f - 2T_f &= \tfrac{1}{2}\big(4 - E_f + \sum_{v\in G_f}(E_v-4)\big) \end{aligned}$$

we have

$$\|G^J\|_\infty \le \mathrm{const}^{n+L}\, M^{\frac{1}{2}j(4-E_\phi)} \prod_{\substack{f\in t\\ f>\phi}} M^{\frac{1}{2}(j_f-j_{\pi(f)})(4-E_f)}$$

The scale sums are performed by repeatedly applying

$$\sum_{\substack{j_f\\ j_f>j_{\pi(f)}}} M^{\frac{1}{2}(j_f-j_{\pi(f)})(4-E_f)} \le \begin{cases} |j| & \text{if } E_f=4\\ \frac{1}{M-1} & \text{if } E_f>4\end{cases} \tag{II.2}$$

starting with the highest forks. This gives

$$\sum_{J\in\mathcal{J}(j,t,R,G)} \|G^J\|_\infty \le \mathrm{const}^{n+L}\, |j|^{L-1} M^j$$

since

$$\#\{f\in t(G^J),\ f\ne\phi\}\le L-1$$

case $D=s'=0,\ s\ne 0$. The modification to handle ∂_p^s is easy. Repeat the eight steps of the previous case, but apply ∂_p^s, in step (2). In the event that $\partial_p^s=\partial_{p_0}$ and $\beta<\infty$, use the discrete product rule

$$\partial_{p_0}\prod_{i=1}^{n+L} f_i(p_0) = \sum_{\ell=1}^{n+L}\prod_{i<\ell} f_i(p_0)\ \partial_{p_0} f(p_0)\ \prod_{i>\ell} f_i\big(p_0+\tfrac{2\pi}{\beta}\big)$$

Application of either product rule can generate at most n_L terms. The external momenta may only appear in vertices, assumed to be C^1 and in propagators, all of scale at least j. The bounds on the L^∞ and L^1 norms of a differentiated propagator given by Lemma II.1 are $\mathrm{const}\, M^{-j}$ times the corresponding norms for an undifferentiated propagator.

case $D=s=0,\ s'\ne 0$. Express $G(p,\beta=\infty)$ as an integral in the usual way. We may express $G^J(p,\beta)$ as the same integral but with the zero component of every loop momentum k_ℓ appearing in the integrand replaced by $[k_{\ell,0}]_\beta$. Hence, using the "product rule"

$$\prod_{i=1}^{n+L} f_i(\beta) - \prod_{i=1}^{n+L} f_i(\infty) = \sum_{i=1}^{n+L}\prod_{m<i} f_m(\infty)\big[f_i(\beta)-f_i(\infty)\big]\prod_{m>i} f_m(\beta) \tag{II.3}$$

the difference $G^J(p,\beta)-G^J(p,\infty)$ can be written as a sum of $n+L$ terms with each term containing one of the differences

$$\left|C^{(j)}\Big(\sum_{i=1}^{n'}\pm k_i\Big) - C^{(j)}\Big(\sum_{i=1}^{n'}\pm([k_{i,0}]_\beta,\mathbf{k})\Big)\right| \le \mathrm{const}\,\frac{n'}{\beta} M^{-2j}$$

$$\left|u_v\Big(\vec{k}_v,\infty\Big) - u_v\Big([\vec{k}_v]_\beta,\beta\Big)\right| \le \mathrm{const}_v\,\frac{1}{\beta}$$

Here, $[\vec{k}_v]_\beta$ designates that, for each line ℓ hooked to v, $k_\ell = \sum_{i=1}^{n'} \pm k_i$ is replaced by $\sum_{i=1}^{n'} \pm([k_{i,0}]_\beta, \mathbf{k}_i)$. So, again, application of ∂_T costs $\mathrm{const}\, M^{-j}$.

case $D>0,\ s=s'=0$. Now G_f is allowed to be two–legged and G is allowed to contain two–legged vertices. Decompose the tree t into a pruned tree $\tilde{t}$ and insertion subtrees $\tau^1, \cdots, \tau^m$ by cutting the branches beneath all minimal $E_f = 2$ forks $f_1, \cdots, f_m$. In other words each of the forks $f_1, \cdots, f_m$ is an $E_f = 2$ fork having no $E_f = 2$ forks, except ϕ, below it in t. Each τ_i consists of the fork f_i and all of t that is above f_i. It has depth at most $D-1$ so the corresponding subgraph G_{f_i} obeys the conclusion of this Proposition. Think of each subgraph G_{f_i} as a generalized vertex in the graph $\tilde{G} = G/\{G_{f_1}, \cdots, G_{f_m}\}$. Thus $\tilde{G}$ now has two as well as four and more–legged vertices. These two–legged vertices have kernels of the form

$$T_i(k) = \sum_{j_{f_i} \le j_{\pi(f_i)}} \ell G_{f_i}(k)$$

when f_i is a c–fork and of the form

$$T_i(k) = \sum_{j_{f_i} > j_{\pi(f_i)}} (\mathbb{1} - \ell) G_{f_i}(k)$$

or

$$T_i(k) = (\mathbb{1} - \ell) u_v(k)$$

when f_i is an r–fork. At least one of the external lines of G_{f_i} must be of scale precisely $j_{\pi(f_i)}$ so the momentum k passing through G_{f_i} lies in the support of $C^{(j_{\pi(f_i)})}$. In the case of a c–fork we have, by Lemma II.3 and the inductive hypothesis

$$\begin{aligned}
\sum_{j_f \le j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} \sup_k \left| \ell G_f^{J_f}(k) \right| &\le \sum_{j_f \le j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} \sup_k \left| G_f^{J_f}(k) \right| \\
&\le \sum_{j_f \le j_{\pi(f)}} \mathrm{const}_{n_f, L_f} |j_f|^{L_f} M^{j_f} \\
&\le \mathrm{const}_{n_f, L_f} M^{j_{\pi(f)}} \sum_{i \ge 0} (|j_{\pi(f)}| + i)^{L_f} M^{-i} \\
&\le \mathrm{const}_{n_f, L_f} |j_{\pi(f)}|^{L_f} M^{j_{\pi(f)}} \sum_{i \ge 0} (i+1)^{L_f} M^{-i} \\
&\le \mathrm{const}_{n_f, L_f} |j_{\pi(f)}|^{L_f} M^{j_{\pi(f)}} \qquad \text{(II.4C)}
\end{aligned}$$

Here t_f and R_f are the restrictions of t and R respectively to forks $f' \geq f$. Hence J_f runs over all assignments of scales to the lines of G_f consistent with the original t and R and with the specified value of j_f. In the case of an r–fork we have, by Lemma II.3 and the inductive hypothesis, the bound

$$\begin{aligned}
&\sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} \sup_k \mathbb{1}\big(C^{(j_{\pi(f)})}(k) \neq 0\big) \left|(\mathbb{1} - \ell) G_f^{J_f}(k, \beta)\right| \\
&\qquad \leq \text{const} \sum_{j_f > j_{\pi(f)}} \sum_{J_f \in \mathcal{J}(j_f, t_f, R_f, G_f)} \left[\tfrac{1}{\beta} \left| \partial_T G_f^{J_f}(k) \right| + M^{j_{\pi(f)}} \sup_k \max_{|\alpha|=1} \left| \partial_k^\alpha G_f^{J_f}(k) \right| \right] \\
&\qquad \leq \text{const}_{n_f, L_f} \, |j_{\pi(f)}|^{L_f - 1} M^{j_{\pi(f)}} \sum_{j_f > j_{\pi(f)}} 1 \\
&\qquad \leq \text{const}_{n_f, L_f} \, |j_{\pi(f)}|^{L_f} M^{j_{\pi(f)}}
\end{aligned} \tag{II.4R}$$

To develop the desired bound we use the basic strategy of the case $D = 0$ with the obvious modifications. In outline that strategy is

(1) Choose a spanning tree $\tilde{T}$ for $\tilde{G}$ with the property that $\tilde{T} \cap \tilde{G}_f^J$ is connected for each $f \in t(\tilde{G}^J)$.

(2) is not used in this case.

(3) Apply (II.4C,R) to bound the suprema, in momentum space, of the two–legged subgraphs $G_{f_1}, \cdots, G_{f_m}$ corresponding to minimal $E_f = 2$ forks.

(4) Bound all remaining vertex functions (all of whom are u_v's) by their suprema in momentum space.

(5) is not used.

(6) Bound all propagators in $\tilde{T}$ by their suprema, using Lemma II.1a.

(7) is not used.

(8) Integrate over the remaining loop momenta. Each remaining loop momentum now just appears in a single factor - the propagator of the unique line of the loop not in $\tilde{T}$. Integration over each remaining loop gives the L^1 norm of that propagator, which has been bounded in Corollary II.2.

The above eight steps give

$$\|G^J\|_\infty \leq \text{const}^{\tilde{L}} \prod_{\substack{v \in \tilde{G} \\ E_v \geq 4}} \|u_v\|_\infty \prod_{i=1}^m \text{const}_{n_{f_i}, L_{f_i}} \, |j|^{L_{f_i}} M^{j_{\pi(f_i)}} \prod_{\ell \in \tilde{T}} M^{-j_\ell} \prod_{\ell \in \tilde{G}^J \setminus \tilde{T}} M^{j_\ell}$$

We again apply

$$M^{\alpha j_\ell} = M^{\alpha j_\phi} \prod_{\substack{f\in\tilde t \\ f>\phi \\ \ell\in\tilde G_f}} M^{\alpha(j_f - j_{\pi(f)})} \tag{II.5a}$$

$$\begin{aligned} 1 &\le M^{-\frac12(E_v-4)j_{\pi(v)}} \qquad\qquad if\ E_v \ge 4 \\ &= M^{-\frac12(E_v-4)j_\phi} \prod_{\substack{f\in\tilde t \\ f>\phi \\ v\in\tilde G_f}} M^{-\frac12(E_v-4)(j_f-j_{\pi(f)})} \end{aligned} \tag{II.5b}$$

to each $M^{\pm j_\ell}$ and $v \in \tilde G$ for which $E_v \ge 4$. In other words (II.5b) is applied for each generalized vertex of $\tilde G$ except for $f_1, \cdots, f_m$. For the latter, we apply

$$\begin{aligned} M^{j_{\pi(f_i)}} &= M^{j_\phi} \prod_{\substack{f\in\tilde t \\ \phi<f<f_i}} M^{j_f - j_{\pi(f)}} \\ &= M^{-\frac12(E_{f_i}-4)j_\phi} \prod_{\substack{f\in\tilde t \\ f>\phi \\ f_i\in\tilde G_f}} M^{-\frac12(E_{f_i}-4)(j_f-j_{\pi(f)})} \end{aligned} \tag{II.5c}$$

which has the same right hand side as (II.5b) would if it applied to the generalized vertex f_i. Application of (II.5a,b,c) gives

$$\|G^J\|_\infty \le \text{const}\, M^{j(\tilde L_\phi - 2\tilde T_\phi - \sum_{v\in\tilde G} \frac12(E_v-4))} \prod_{i=1}^m |j|^{L_{f_i}} \prod_{\substack{f\in\tilde t \\ f>\phi}} M^{(j_f-j_{\pi(f)})(\tilde L_f - 2\tilde T_f - \sum_{v\in\tilde G_f} \frac12(E_v-4))}$$

where

$$\tilde T_f = \text{number of lines of } \tilde T \cap \tilde G_f$$

$$\tilde L_f = \text{number internal lines of } \tilde G_f$$

The sums $\sum_{v\in\tilde G}$ and $\sum_{v\in\tilde G_f}$ run over two– as well as four– and more–legged generalized vertices. Hence, again,

$$\begin{aligned} \tilde L_f &= \tfrac12\big(\sum_{v\in\tilde G_f} E_v - E_f\big) \\ \tilde T_f &= \sum_{v\in\tilde G_f} 1 - 1 \\ \Longrightarrow \tilde L_f - 2\tilde T_f &= \tfrac12\big(4 - E_f + \sum_{v\in\tilde G_f}(E_v-4)\big) \end{aligned}$$

and we have

$$\|G^J\|_\infty \le \text{const}_{n,L}\, M^j \prod_{i=1}^m |j|^{L_{f_i}} \prod_{\substack{f\in\tilde t\\ f>\phi}} M^{\frac{1}{2}(j_f-j_{\pi(f)})(4-E_f)}$$

The scale sums are performed by repeatedly applying (II.2) as in the case D=0 and give at most $\tilde L-1$ factors of j. The desired bound follows from

$$\tilde L - 1 + \sum_{i=1}^m L_{f_i} = L-1$$

case $D>0,\ s\neq 0$ **or** $s'\neq 0$. Prune t and prepare $\tilde G$ as in the case $D>0,\ s=s'=0$. As in the case $D=0$, the derivative ∂_p or ∂_T may act on four or more–legged vertices and on propagators of $\tilde G$, all of scale at least j. In the former case the derivative costs at most const because the vertex functions are assumed to be C^1. In the latter case the derivative costs at most const M^{-j}, by Lemma II.1 b).

But now, unlike the case $D=0$, the derivative may also act on a two–legged generalized vertex. This vertex is necessarily either a c–fork or an r–fork. Recall that the value of a c–fork is constructed by applying the localization operator ℓ to some two–legged diagram. This localization operator evaluates at $\beta=\infty$. So the value of the c–fork is independent of β. By Lemma II.3b, ∂_p costs at most $M^{-j_{\pi(f_i)}}\le M^{-j}$. Now consider an r–fork. Application of ∂_p gives a bound, by Lemma II.3d, that is at most a factor of M^{-j} larger than the $D=0$ bound (II.4R). As for the effect of ∂_T, (II4.R) is replaced by

$$\begin{aligned}
&\sum_{j_f>j_{\pi(f)}}\ \sum_{J_f\in\mathcal{J}(j_f,t_f,R_f,G_f)} \sup_k \mathbb{1}\big(C^{j_{\pi(f)}}(k)\neq 0\big)\left|\partial_T(\mathbb{1}-\ell)G_f^{J_f}(k,\beta)\right|\\
&\quad=\sum_{j_f>j_{\pi(f)}}\ \sum_{J_f\in\mathcal{J}(j_f,t_f,R_f,G_f)}\left|\partial_T G_f^{J_f}(k,\beta)\right|\\
&\quad\le \text{const}_{n_f,L_f}\,|j|^{L_f-1}\sum_{j_f>j_{\pi(f)}} 1\\
&\quad\le \text{const}_{n_f,L_f}\,|j|^{L_f}
\end{aligned}$$

which is again no more than a factor of M^{-j} worse than (II.4R). ■

Corollary II.5 *Assume Hypotheses H1,2,3',4'. Let $G(p,\beta)$ be any graph contributing to the proper self–energy. Then, for every $0 \le \epsilon < 1$,*

$$\sup_{p,\beta} |G(p,\beta)| < \infty$$
$$\sup_{p,\beta} \beta^\epsilon |G(p,\beta) - G(p,\infty)| < \infty$$

Proof: The first bound is immediate from Proposition II.4 with $s = 0$. One merely has to sum over j, t and R. For the second bound, first prove

$$\sum_{J \in \mathcal{J}(j,t,R,G)} \sup_{p,\beta} \beta^\epsilon \left| G^J(p,\beta) - G^J(p,\infty) \right| \le \text{const}_{n,L} \, |j|^{L-1} M^{j(1-\epsilon)}$$

by taking a convex combination of the two bounds of Proposition II.4. Then sum over j, t and R. ■

§II.4 Bounds on general graphs

In the last section we showed that graphs contributing to the proper self-energy were Hölder continuous in $1/\beta$ at $\beta = \infty$ for all indices ϵ with $0 \le \epsilon < 1$. The graphs, as functions of their external momentum, were viewed as elements of L^∞. We now use both the method and the results of the last section to show that general graphs, viewed as L^1 functions of their external momenta, are also Hölder continuous in $1/\beta$ at $\beta = \infty$.

Proposition II.6 *Assume Hypotheses H1,2,3',4'. Let $G(\vec{p},\beta)$ be any graph contributing to the E–point connected Euclidean Green's functions. Then, for every $0 \le \epsilon < 1$,*

$$\sup_{\beta} \|G(\,\cdot\,,\beta)\|_1 < \infty$$
$$\sup_{\beta} \beta^\epsilon \|G(\,\cdot\,,\beta) - G(\,\cdot\,,\infty)\|_1 < \infty$$

Proof: We apply the argument of the last section to an extension G^* of G. The extension is chosen to have two external legs. It is also chosen so that its value is independent of the external momentum and equals the desired L^1 norm of G. This enables us to repeat the argument of the last section almost verbatim. The vertices of G^* are the vertices of G plus one extra vertex v^*. All the vertices of G^*, with the exception of v^* are internal. The internal lines of G^* are all of the internal and external lines of G. The external lines of G are viewed as internal lines of G^*. They are all hooked to v^*. The external lines of G^* consist of two lines, both emanating from v^*. The construction automatically results in integration over the external momenta subject to global conservation of momentum. To compute $\sup_\beta \|G(\,\cdot\,,\beta)\|_1$ we choose the vertex function of v^* to be $\overline{G(\vec{p},\beta)}/|G(\vec{p},\beta)|$. To compute $\beta\|G(\,\cdot\,,\beta)-G(\,\cdot\,,\infty)\|_1$, for some fixed value β_0 of β, we choose the vertex function of v^* to be $\overline{G(\vec{p},\beta_0)-G(\vec{p},\infty)}/|G(\vec{p},\beta_0)-G(\vec{p},\infty)|$. We now bound G^*. The rest of the prood is similar to that of Proposition II.3. It is included for completeness.

We again use (II.1r)

$$G^* = \sum_{t\in\mathcal{F}(G^*)} \sum_{\substack{R\in\mathcal{R}(t)\\ R_\phi=c}} \sum_{j\le 0} \sum_{J\in\mathcal{J}(j,t,R,G^*)} G^{*J}$$

to block the contributions to G^* according to the scales of the various propagators. We also decompose the tree t into a pruned tree $\tilde{t}$ and insertion subtrees $\tau^1,\cdots,\tau^m$ by cutting the branches beneath all minimal $E_f=2$ forks $f_1,\cdots,f_m$. Each of the forks $f_1,\cdots,f_m$ is an $E_f=2$ fork having no $E_f=2$ forks (other than ϕ) below it in t. Because v^* itself has two external legs, no $G^*_{f_i}$ may contain v^*. Each τ_i consists of the fork f_i and all of t that is above f_i. The corresponding subgraph $G^*_{f_i}$ obeys the conclusion of Proposition II.4. Think of each subgraph $G^*_{f_i}$ as a generalized vertex in the graph $\tilde{G}^*=G^*/\{G^*_{f_1},\cdots,G^*_{f_m}\}$. Yet again,

(1) Choose a spanning tree $\tilde{T}$ for $\tilde{G}^*$ with the property that $\tilde{T}\cap G^*_f$ is connected for each $f\in t(\tilde{G}^{*J})$.

(2) apply ∂_T, using the product rule, if appropriate. Note that, by construction, the ∂_T may not act on u_{v^*}.

(3) Apply (II.4C,R) to bound the suprema, in momentum space, of the two–legged subgraphs $G_{f_1},\cdots,G_{f_m}$ corresponding to minimal $E_f=2$ forks. Note that the bound on G_{f_i} includes the sum over all scales $j_{f'}$ with $f'\ge f_i$. Also note that no

G_{f_i} contains v^*, so no momentum derivative (arising during renormalization) ever acts on u_{v^*}.

(4) Take the supremum of all remaining vertex functions u_v. In particular, bound $|u_{v^*}| = 1$.

(5) is not used.

(6) Bound all propagators in $\tilde{T}$ using $\left\|C^{(j_\ell)}\right\|_\infty \le M^{-j_\ell}$.

(7) is not used.

(8) Each loop momentum k_ℓ now appears in precisely one factor of $|C^{(j_\ell)}(k_\ell)|$ and nowhere else in the integrand. Integrate over the remaining loop momenta. Integration over k_ℓ gives an L^1 norm of $C^{(j_\ell)}$, which has been bounded in Corollary II.2.

The above eight steps give

$$\sum_{\substack{j_{f'} \\ f' \ge f_i \text{ for} \\ \text{some } 1 \le i \le m}} \|G^J\|_1 \le \prod_v \|u_v\|_\infty \prod_{\ell \in \tilde{T}} M^{-j_\ell} \prod_{\ell \in G^{*J} \setminus \tilde{T}} \text{const}\, M^{j_\ell}$$
$$\prod_{i=1}^m \text{const}_{n_{f_i}, L_{f_i}} \left|j_{\pi(f_i)}\right|^{L_{f_i}} M^{j_{\pi(f_i)}}$$

They also give

$$\sum_{\substack{j_{f'} \\ f' \ge f_i \text{ for} \\ \text{some } 1 \le i \le m}} \|\partial_T G^J\|_1 \le \prod_{\ell \in \tilde{T}} \text{const}\, M^{-j_\ell} \prod_{\ell \in G^{*J} \setminus \tilde{T}} \text{const}\, M^{j_\ell} \prod_{i=1}^m \text{const}_{n_{f_i}, L_{f_i}} \left|j_{\pi(f_i)}\right|^{L_{f_i}} M^{j_{\pi(f_i)}}$$
$$\left\{ M^{-j_\phi} \prod_v \|u_v\|_\infty + \sum_v \|(\partial_T + \partial_{\vec{k}_v}) u_v\|_\infty \prod_{v' \ne v} \|u_{v'}\|_\infty \right\}$$

Yet again, we apply

$$M^{\alpha j_\ell} = M^{\alpha j_\phi} \prod_{\substack{f \in \tilde{t} \\ f > \phi \\ \ell \in \tilde{G}^*_f}} M^{\alpha (j_f - j_{\pi(f)})} \tag{II.6a}$$

$$1 \le M^{-\frac{1}{2}(E_v - 4) j_\phi} \prod_{\substack{f \in \tilde{t} \\ f > \phi \\ v \in \tilde{G}^*_f}} M^{-\frac{1}{2}(E_v - 4)(j_f - j_{\pi(f)})} \qquad \text{if } E_v \ge 4 \tag{II.6b}$$

$$M^{j_{\pi(f_i)}} = M^{-\frac{1}{2}(E_{f_i} - 4) j_\phi} \prod_{\substack{f \in \tilde{t} \\ f > \phi \\ f_i \in \tilde{G}^*_f}} M^{-\frac{1}{2}(E_{f_i} - 4)(j_f - j_{\pi(f)})} \tag{II.6c}$$

to each $M^{\pm j_\ell}$, $v \in \tilde G^*$ for which $E_v \ge 4$ and $1 \le i \le m$. Note that $E_{v^*} \ge 4$. Application of (II.6a,b,c) gives

$$\begin{aligned}\sum_{\substack{j_{f'}\\ f'\ge f_i \text{ for}\\ \text{some } 1\le i\le m}} \|G^J\|_1 &\le \text{const} \prod_{i=1}^m |j|^{L_{f_i}} M^{j(\tilde L_\phi - 2\tilde T_\phi - \sum_{v\in\tilde G^*} \frac12 (E_v-4))} \\ &\qquad \prod_{\substack{f\in\tilde t\\ f>\phi}} M^{(j_f - j_{\pi(f)})(\tilde L_f - 2\tilde T_f - \sum_{v\in \tilde G_f^*} \frac12(E_v-4))} \\ &\le \text{const}_{n,L}\, M^j \prod_{i=1}^m |j|^{L_{f_i}} \prod_{\substack{f\in\tilde t\\ f>\phi}} M^{\frac12 (j_f - j_{\pi(f)})(4-E_f)} \end{aligned} \tag{II.7}$$

where

$$\tilde T_f = \text{number of lines of } \tilde T \cap \tilde G_f^*$$

$$\tilde L_f = \text{number internal lines of } \tilde G_f^*$$

We have used that G^* has two external legs so that

$$M^{\frac12 j(4-E_\phi)} = M^j$$

Similarly

$$\sum_{\substack{j_{f'}\\ f'\ge f_i \text{ for}\\ \text{some } 1\le i\le m}} \|G^J(\,\cdot\,,\beta) - G^J(\,\cdot\,,\infty)\|_1 \le \text{const}_{n,L} \frac{1}{\beta} \prod_{i=1}^m |j|^{L_{f_i}} \prod_{\substack{f\in\tilde t\\ f>\phi}} M^{\frac12 (j_f - j_{\pi(f)})(4-E_f)} \tag{II.8}$$

For the bound on $\sup_\beta \|G(\,\cdot\,,\beta)\|_1$, the scale sums are performed by repeatedly applying (II.2) as usual. For the bound on $\sup_\beta \|G(\,\cdot\,,\beta)-G(\,\cdot\,,\infty)\|_1$, we use a geometric mean between (II.8) and (II.7) applied twice (on $\sum \|G^J(\,\cdot\,,\beta) - G^J(\,\cdot\,,\infty)\|_1 \le \sum \|G^J(\,\cdot\,,\beta)\|_1 + \sum \|G^J(\,\cdot\,,\infty)\|_1$) to get

$$\sum_{\substack{j_{f'}\\ f'\ge f_i \text{ for}\\ \text{some } 1\le i\le m}} \|G^J(\,\cdot\,,\beta) - G^J(\,\cdot\,,\infty)\|_1 \le \text{const}_{n,L} \frac{1}{\beta^\epsilon} M^{(1-\epsilon)j} \prod_{i=1}^m |j|^{L_{f_i}} \prod_{\substack{f\in\tilde t\\ f>\phi}} M^{\frac12 (j_f - j_{\pi(f)})(4-E_f)}$$

and then perform the scale sums by repeatedly applying (II.2). ■

§III. The Ultraviolet End

In this section, we show that one may remove the ultraviolet cutoff in the k_0 direction uniformly in β. The overall strategy is as follows. Fix any graph. The propagator is first written as the sum of an infrared part and an ultraviolet part. The "full" model has propagator

$$C(k) = \frac{\rho(|\mathbf{k}|/\mathfrak{C})}{ik_0 - e(\mathbf{k})}$$

with $\rho(|\mathbf{k}|/\mathfrak{C})$ providing a fixed ultraviolet cutoff in spatial directions only. For this chapter, we assume that there is an $r \geq 0$ such that $e(\mathbf{k})$ obeys

(H1uv) $e(\mathbf{k})$ is C^r

The infrared (ir) propagator is

$$\sum_{j\leq 0} C^{(j)}(k) = \delta_{\sigma,\sigma'} \frac{\rho(k_0^2 + e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})}$$

We may assume, without loss of generality, that $\lim_{\mathbf{k}\to\infty} e(\mathbf{k}) = \infty$. We are also assuming that $\mathfrak{C}$ is sufficiently large that if $\rho(k_0^2 + e(\mathbf{k})^2)$ is nonzero, then $\rho(|\mathbf{k}|/\mathfrak{C}) = 1$. So the uv propagator is

$$U(k) = \delta_{\sigma,\sigma'} \frac{h(k_0^2 + e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})} \rho(|\mathbf{k}|/\mathfrak{C})$$

with $h(k_0^2 + e(\mathbf{k})^2) = 1 - \rho(k_0^2 + e(\mathbf{k})^2)$ forcing $k_0^2 + e(\mathbf{k})^2 >$ const > 0.

Vertex functions need only be bounded and suitably smooth in momentum space. They need not decay at infinity. For example a delta function two–body interaction is perfectly acceptable. We shall actually work in a mixed time/spatial momentum space. The precise hypotheses on the vertex functions are

(H3uv) Every interaction vertex has an even number of external (particle) legs.

(H4uv) The vertex function $(2\pi)^d \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_{2q}) U_v(t_1, \mathbf{k}_1, \cdots, t_{2q}, \mathbf{k}_{2q})$ associated with each $2q$–legged vertex v has U_v obeying

$$\max_{1\leq j\leq 2q} \sup_{\substack{\mathbf{k}_1,\cdots,\mathbf{k}_{2q} \\ t_j}} \int \Big| \prod_{i=1}^{2q} \partial_{\mathbf{k}_i}^{\alpha_i} \prod_{i=1}^{2q} [1 + |t_i - t_j|]^{N_i} U_v(t_1, \mathbf{k}_1, \cdots, t_{2q}, \mathbf{k}_{2q}) \Big| \prod_{\substack{i=1 \\ i\neq j}}^{2q} dt_i < \infty$$

for all $\sum_i N_i \leq r + 2$ and $|\alpha| \leq r$.

To bound the full graph, it suffices to bound the contribution to the value of the graph arising from an arbitrary but fixed assignment of ir/uv propagator to each line of the graph. The subgraph consisting of all lines to which the uv part of the propagator has been assigned is a finite union of connected components. We think of these components as generalized vertices in a graph containing only ir propagators. In this chapter, we prove bounds on these generalized vertices. We have already treated graphs having only ir propagators in §II and we will develop additional bounds on them in [FKST1]. We now prove

Proposition III.1 *Let G be any connected graph of order n having only uv propagators. Denote by $G_\beta(k)$ its value in momentum space, excluding the conservation of momentum delta function. If G has $2q$ external legs, k runs over $\left(\frac{\pi}{\beta}(2\mathbb{Z}+1)\times\mathcal{D}\right)^{2q-1}$. Assume (H1uv), (H3uv) and (H4uv). Then,*

$$\sup_{\beta\le\infty}\sup_{|\alpha|\le r}\sup_{k}|\partial_k^\alpha G_\beta(k)|\le \mathrm{const}(r)^n$$

$$\sup_{|\alpha|\le r}\sup_{k}|\partial_k^\alpha G_\beta(k)-\partial_k^\alpha G_\infty(k)|\le\frac{\mathrm{const}(r)^n}{\beta}$$

with the constant const(r) *independent of β. As usual α is a multiindex and ∂^α a difference/differential operator.*

We shall actually prove bounds in a mixed time/momentum space. The zero component of each $(d+1)$–vector $(t,\mathbf{k})$ denotes a Euclidean time. The other components denote a momentum. The norms of Proposition III.1 are bounded by taking L^1 norms in temporal directions and k_0–derivatives are implemented by multiplying by differences of external temporal arguments. We use the Poisson summation formula to write $U_\beta(t,\mathbf{k})$ as a sum of $U(t,\mathbf{k})$ plus terms that vanish as $\beta\to\infty$. A variant of the classical Poisson summation formula is

Lemma III.2 *Let $\hat f\in C^2(\mathbb{R})$ and*

$$\left|\hat f(k_0)\right|+\left|\hat f'(k_0)\right|+\left|\hat f''(k_0)\right|\le\frac{\mathrm{const}}{1+k_0^2}$$

Set

$$f(t)=\int \hat f(k_0)e^{-ik_0t}\,dk_0$$

Then

$$\tfrac{1}{\beta}\sum_{k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\hat{f}(k_0)e^{-ik_0t}=\sum_{n\in\mathbb{Z}}(-1)^n f(t-n\beta)$$

Proof: By the Poisson summation formula [DM,§2.7.5],

$$\tfrac{1}{L}\sum_{k_0\in\frac{2\pi}{L}\mathbb{Z}}\hat{f}(k_0)e^{-ik_0t}=\sum_{\tau\in L\mathbb{Z}}f(t-\tau)$$

First choosing $L=2\beta$

$$\tfrac{1}{\beta}\sum_{k_0\in\frac{\pi}{\beta}\mathbb{Z}}\hat{f}(k_0)e^{-ik_0t}=2\sum_{\tau\in 2\beta\mathbb{Z}}f(t-\tau)$$

and then choosing $L=\beta$

$$\tfrac{1}{\beta}\sum_{k_0\in\frac{2\pi}{\beta}\mathbb{Z}}\hat{f}(k_0)e^{-ik_0t}=\sum_{\tau\in \beta\mathbb{Z}}f(t-\tau)$$

and finally subtracting and using $\frac{\pi}{\beta}(2\mathbb{Z}+1)=\frac{\pi}{\beta}\mathbb{Z}\setminus\frac{\pi}{\beta}2\mathbb{Z}$

$$\begin{aligned}\tfrac{1}{\beta}\sum_{k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\hat{f}(k_0)e^{-ik_0t}&=\sum_{\tau\in\beta\mathbb{Z}}f(t-\tau)\begin{cases}2-1&\text{if }\tau\in 2\beta\mathbb{Z}\\-1&\text{otherwise}\end{cases}\\&=\sum_{n\in\mathbb{Z}}(-1)^n f(t-n\beta)\end{aligned}$$

■

Fix any $\mathbf{k}$ with $e(\mathbf{k})\neq 0$. The function $\hat{f}(k_0)=\frac{1}{ik_0-e(\mathbf{k})}$ does not satisfy the hypotheses of the previous Lemma. In fact it would not make sense if it did because the Fourier transform of $\hat{f}(k_0)=\frac{1}{ik_0-e(\mathbf{k})}$ is not continuous at $t=0$. Its value at $t=0$ is defined by the limit $t\nearrow 0$. By definition

$$\begin{aligned}C(t,\mathbf{k})=\int\frac{dk_0}{2\pi}\frac{e^{-ik_0(t-0)}}{ik_0-e(\mathbf{k})}&=e^{-e(\mathbf{k})t}\begin{cases}-\chi\big(e(\mathbf{k})>0\big)&\text{if }t>0\\ \chi\big(e(\mathbf{k})<0\big)&\text{if }t\le 0\end{cases}\\&=e^{-e(\mathbf{k})t}\begin{cases}-\chi\big(t>0\big)&\text{if }e(\mathbf{k})>0\\ \chi\big(t\le 0\big)&\text{if }e(\mathbf{k})<0\end{cases}\end{aligned}$$

and, with the standard notation $n_{\mathbf{k}}=\left[e^{\beta e(\mathbf{k})}+1\right]^{-1}$,

$$C_\beta(t,\mathbf{k})=\tfrac{1}{\beta}\sum_{k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)}\tfrac{1}{ik_0-e(\mathbf{k})}e^{-ik_0(t-0)}=e^{-e(\mathbf{k})t}\begin{cases}-1+n_{\mathbf{k}}&\text{if }0<t<\beta\\ n_{\mathbf{k}}&\text{if }-\beta<t\le 0\end{cases}$$

Again $C_\beta(t,\mathbf{k})$ is really defined at $t=0$ via the limit $t\nearrow 0$.

Lemma III.3 *Let* $\mathbf{k}$ *be such that* $e(\mathbf{k}) \neq 0$.

$$C_\beta(t,\mathbf{k}) = \sum_{m\in\mathbb{Z}} (-1)^m C(t-\beta m,\mathbf{k})$$

The right hand side converges absolutely.

Proof: If $e(\mathbf{k}) < 0$

$$n_{\mathbf{k}} = \sum_{n=0}^{\infty} (-1)^n e^{n\beta e(\mathbf{k})}$$

and if $e(\mathbf{k}) > 0$

$$n_{\mathbf{k}} = \frac{e^{-\beta e(\mathbf{k})}}{1+e^{-\beta e(\mathbf{k})}} = \sum_{n=0}^{\infty} (-1)^n e^{-(n+1)\beta e(\mathbf{k})}$$

so that, for $-\beta < t < \beta$,

$$\begin{aligned}
C_\beta(t,\mathbf{k}) &= e^{-e(\mathbf{k})t} \begin{cases} -1+n_{\mathbf{k}} & \text{if } t>0 \\ n_{\mathbf{k}} & \text{if } t\le 0 \end{cases} \\
&= e^{-e(\mathbf{k})t} n_{\mathbf{k}} - \begin{cases} e^{-e(\mathbf{k})t} & \text{if } t>0 \\ 0 & \text{if } t\le 0 \end{cases} \\
&= \sum_{n=0}^{\infty} (-1)^n e^{-e(\mathbf{k})t} \left\{ \begin{matrix} e^{-(n+1)\beta e(\mathbf{k})} & e(\mathbf{k})>0 \\ e^{n\beta e(\mathbf{k})} & e(\mathbf{k})<0 \end{matrix} \right\} - \begin{cases} e^{-e(\mathbf{k})t} & t>0 \\ 0 & t\le 0 \end{cases} \\
&= \sum_{m\in\mathbb{Z}} (-1)^m e^{-e(\mathbf{k})(t-m\beta)} \left\{ \begin{matrix} -\chi(m<0) & e(\mathbf{k})>0 \\ \chi(m\ge 0) & e(\mathbf{k})<0 \end{matrix} \right\} - \begin{cases} e^{-e(\mathbf{k})t} & t>0 \\ 0 & t\le 0 \end{cases} \\
&= \sum_{m\neq 0} (-1)^m e^{-e(\mathbf{k})(t-m\beta)} \left\{ \begin{matrix} -\chi(m<0) & e(\mathbf{k})>0 \\ \chi(m> 0) & e(\mathbf{k})<0 \end{matrix} \right\} \\
&\qquad + e^{-e(\mathbf{k})t}\chi\big(e(\mathbf{k})<0\big) - \begin{cases} e^{-e(\mathbf{k})t} & t>0 \\ 0 & t\le 0 \end{cases} \\
&= \sum_{m\neq 0} (-1)^m e^{-e(\mathbf{k})(t-m\beta)} \left\{ \begin{matrix} -\chi(t-\beta m>0) & e(\mathbf{k})>0 \\ \chi(t-\beta m< 0) & e(\mathbf{k})<0 \end{matrix} \right\} \\
&\qquad + e^{-e(\mathbf{k})t} \begin{cases} -\chi(t>0) & e(\mathbf{k})>0 \\ \chi(t\le 0) & e(\mathbf{k})<0 \end{cases}
\end{aligned}$$

To achieve the last line we used that $-\beta < t < \beta$ and $m \in \mathbb{Z}\setminus\{0\}$ imply $t-\beta m > 0 \iff m < 0$ and $t-\beta m < 0 \iff m > 0$. The right hand side is $\sum_{m\in\mathbb{Z}}(-1)^m C(t-\beta m,\mathbf{k})$ as desired. This verifies the claimed formula for $-\beta < t < \beta$. Both sides of the formula change sign under $t \to t+\beta$. ■

Define

$$U(t,\mathbf{k}) = \delta_{\sigma,\sigma'} \int \frac{dk_0}{2\pi} e^{-ik_0(t-0)} \frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})} \rho(|\mathbf{k}|/\mathfrak{C})$$
$$U_\beta(t,\mathbf{k}) = \delta_{\sigma,\sigma'} \tfrac{1}{\beta} \sum_{k_0 \in \frac{\pi}{\beta}(2\mathbb{Z}+1)} e^{-ik_0(t-0)} \frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})} \rho(|\mathbf{k}|/\mathfrak{C})$$

Lemma III.4 *a)*

$$U_\beta(t,\mathbf{k}) = \sum_{m\in\mathbb{Z}} (-1)^m U(t-\beta m, \mathbf{k})$$

b) Let $e(\mathbf{k}) \in C^r$. Then, for all $|\alpha| \le r$ and $n \ge 0$

$$\sup_{\beta\ge 1} \sup_{\substack{|t|\le\beta/2 \\ \mathbf{k}\in\mathbb{R}^d}} (1+|t|)^n \left|\partial_{\mathbf{k}}^\alpha U_\beta(t,\mathbf{k})\right| < \infty$$
$$\sup_{\substack{t\in\mathbb{R} \\ \mathbf{k}\in\mathbb{R}^d}} (1+|t|)^n \left|\partial_{\mathbf{k}}^\alpha U(t,\mathbf{k})\right| < \infty$$

Proof: a) For $|e(\mathbf{k})| \ge \frac{1}{2}$, write

$$\frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})} = \frac{1}{ik_0 - e(\mathbf{k})} - \frac{\rho(k_0^2+e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})}$$

The first term is handled by Lemma III.3 and the second by Lemma III.2. For $|e(\mathbf{k})| \le \frac{1}{2}$, write

$$\begin{aligned}\frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})} &= \frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0 - 1} + \frac{[e(\mathbf{k})-1]h(k_0^2+e(\mathbf{k})^2)}{[ik_0 - e(\mathbf{k})][ik_0-1]} \\ &= \frac{1}{ik_0-1} - \frac{\rho(k_0^2+e(\mathbf{k})^2)}{ik_0 - 1} + \frac{[e(\mathbf{k})-1]h(k_0^2+e(\mathbf{k})^2)}{[ik_0 - e(\mathbf{k})][ik_0-1]}\end{aligned}$$

The first term is handled by Lemma III.3 and the second and third by Lemma III.2.

b) The first result follows from the second and part a). Just use two of the powers from $(1+|t|)^n$ to control the sum over m. So we now prove the second result.

Case $|t| \ge 1$. Here we may, without loss of generality, restrict to $n \ge 1$.

$$\begin{aligned}t^n U(t,\mathbf{k}) &= \delta_{\sigma,\sigma'} \int \frac{dk_0}{2\pi} \frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})} \rho(|\mathbf{k}|/\mathfrak{C}) \left(i\frac{d}{dk_0}\right)^n e^{-ik_0 t} \\ &= \delta_{\sigma,\sigma'} \int \frac{dk_0}{2\pi} e^{-ik_0 t} \rho(|\mathbf{k}|/\mathfrak{C}) \left(-i\frac{d}{dk_0}\right)^n \frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k})}\end{aligned}$$

For all $n \geq 1$ and $|\alpha| \leq r$, $\partial_{\mathbf{k}}^{\alpha} \left(-i\frac{d}{dk_0}\right)^n \frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0-e(\mathbf{k})}$ has a finite L^1 norm in k_0 that is bounded uniformly in $\mathbf{k}$, because each $-i\frac{d}{dk_0}$ either acts on the h, restricting k_0 to a compact set, or acts on the $\frac{1}{ik_0-e(\mathbf{k})}$, which increases the power of k_0 downstairs to at least two. Hence

$$\sup_{t,\mathbf{k}} |t^n U(t,\mathbf{k})| < \infty$$

This implies the desired result for $|t| \geq 1$.

Case $|t| \leq 1,\ |e(\mathbf{k})| \geq \frac{1}{2}$. Write

$$\frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0-e(\mathbf{k})} = \frac{1}{ik_0-e(\mathbf{k})} - \frac{\rho(k_0^2+e(\mathbf{k})^2)}{ik_0-e(\mathbf{k})}$$

The Fourier transform (with respect to k_0) of the first term is

$$e^{-e(\mathbf{k})t}\begin{cases} -\chi\big(t>0\big) & \text{if } e(\mathbf{k})>0 \\ \chi\big(t\leq 0\big) & \text{if } e(\mathbf{k})<0 \end{cases}$$

That of the second term, and its first r derivatives in $\mathbf{k}$, is Schwarz class in t.

Case $|t| \leq 1,\ |e(\mathbf{k})| \leq \frac{1}{2}$. Write

$$\frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0-e(\mathbf{k})} = \frac{h(k_0^2+e(\mathbf{k})^2)}{ik_0-1} + \frac{[e(\mathbf{k})-1]h(k_0^2+e(\mathbf{k})^2)}{[ik_0-e(\mathbf{k})][ik_0-1]}$$

The first term is handled as in the case $|t| \leq 1,\ |e(\mathbf{k})| \geq \frac{1}{2}$. The second term is L^1 in k_0. So its Fourier transform in k_0 is L^∞. Multiplying by powers of t is implemented by taking derivatives with respect to k_0. These derivatives and up to r derivatives with respect to $\mathbf{k}$ also give L^1 functions in k_0. The bounds are uniform in $\mathbf{k}$ because $h(k_0^2+e(\mathbf{k})^2)$ forces the denominators to be bounded away from zero. ■

Proof of Proposition III.1: Recall that the value of the graph G in momentum space, when the propagators and interaction are given in time/spatial momentum space is computed as

$$\int \delta(t_{2q}) \prod_{\ell\in\mathcal{L}} U_\beta(t_{v_\ell,i_\ell} - t_{v'_\ell,i'_\ell}, \mathbf{k}_\ell) \prod_{v\in\mathcal{V}} U_v(\vec{t}_v, \vec{\mathbf{k}}_v) \prod_{\ell\in G\setminus T} d\!\!\!{}^-\mathbf{k}_\ell \prod_{j=1}^{2q-1} e^{(-1)^j ik_{j,0}t_j} \prod_{v\in\mathcal{V}} \prod_{i=1}^{2q_v} dt_{v,i}$$

where

- $\mathcal{L}$ is the set of internal lines. Line ℓ joins leg number i_ℓ of vertex v_ℓ to leg number i'_ℓ of vertex v'_ℓ. Spatial momentum loops are specified by choosing a spanning tree T for G as in §I.2. If $\ell \in G \setminus T$, then the spatial momentum $\mathbf{k}_\ell$ flowing through ℓ is itself, by construction, a loop momentum and is integrated over $\mathcal{D}$, the support of $\rho(\mathbf{k}/\mathfrak{C})$. If $\ell \in T$ the spatial momentum k_ℓ flowing through ℓ is the signed sum of all loop and external momenta passing through ℓ.
- $\mathcal{V}$ is the set of vertices. Vertex v has $2q_v$ legs. The $i^{\rm th}$ leg of vertex v has temporal argument $t_{v,i}$ and spatial momentum argument $\mathbf{k}_{v,i}$. The set of all arguments for vertex v is denoted $(\vec{t}_v, \vec{\mathbf{k}}_v)$. The temporal argument $t_{v,i}$ of each leg of each vertex, except the $2q^{\rm th}$ external leg of G, is integrated over $(-\beta/2, \beta/2)$.
- The temporal argument of the $2q^{\rm th}$ external leg of G is held fixed at the origin. The temporal arguments of the external legs of G are denoted $t_1, t_2, \cdots, t_{2q}$ and the momentum arguments are denoted $\mathbf{k}_1, \cdots, \mathbf{k}_{2q-1}$, with the even subscripts associated with incoming external particle lines and the odd ones with outgoing ones. Because all the external lines of G have been amputated, each external leg of G coincides with a leg of some vertex. Thus for each $1 \le j \le 2q$, $t_j = t_{v_j, i_j}$ for some $v_j,\ i_j$.

Because $\mathbf{k}_\ell \in G \setminus T$ is restricted to a fixed compact set, it suffices to prove bounds on

$$\int \delta(t_{2q}) \partial_{\mathbf{k}}^\alpha \prod_{\ell \in \mathcal{L}} U_\beta(t_{v_\ell, i_\ell} - t_{v'_\ell, i'_\ell}, \mathbf{k}_\ell) \prod_{v \in \mathcal{V}} U_v(\vec{t}_v, \vec{\mathbf{k}}_v) \prod_{j=1}^{2q-1} e^{(-1)^j i k_{j,0} t_j} \prod_{v \in \mathcal{V}} \prod_{i=1}^{2q_v} dt_{v,i}$$

pointwise, but uniform, in $\mathbf{k}_\ell \in \mathcal{D},\ \ell \in G \setminus T$.

We first prove that if you apply derivatives $\partial_{\mathbf{k}}$ with respect to the external momenta and multiply the integrand by a monomial in the differences between the temporal arguments of pairs of external arguments, the result is L^1 in the $t_{v,i}$'s, uniformly in β and $\mathbf{k}_\ell$'s. This implies the first bound of the Proposition. To be picky, when $\beta < \infty$, differencing with respect to the temporal component of the $j^{\rm th}$ external momentum corresponds, in position space, to multiplication by $\frac{\beta}{2\pi}\left(e^{-(-1)^j i \frac{2\pi}{\beta} t_j} - 1\right)$ rather than to multiplication by $(-1)^{j+1} i t_j$. However, as

$$\left| \tfrac{\beta}{2\pi}\left(e^{-(-1)^j i \frac{2\pi}{\beta} t_j} - 1\right) \right| \le \text{const}\, |t_j|$$

for all $|t_j| \le \beta/2$, this is immaterial from the point of view of bounds.

Apply the $\partial_{\mathbf{k}}^{\alpha}$, using the product rule to distribute the derivatives amongst the various U_v's and U_β's. If $\beta < \infty$ substitute

$$U_\beta(t, \mathbf{k}) = \sum_{m \in \mathbb{Z}} (-1)^m U\big(t - \beta m, \mathbf{k}\big)$$

for each propagator. The temporal arguments of all vertices obey $|t_{v,i}| < \beta/2$. So the temporal arguments $t = t_{v_\ell, i_\ell} - t_{v'_\ell, i'_\ell}$ of all propagators obey $|t| < \beta$. So we may bound, for any fixed N and any $|\alpha'| \le r$

$$\begin{aligned} |\partial_{\mathbf{k}}^{\alpha'} U\big(t - m\beta, \mathbf{k}\big)| &\le \text{const}\, \frac{1}{[1 + |t - m\beta|]^{N+2}} \\ &\le \text{const}\, \frac{1}{1 + m^2} \frac{1}{[1 + |t - m\beta|]^{N}} \end{aligned}$$

The sum over m for each propagator is easily controlled by the $[1 + m^2]^{-1}$. So we drop all of the $[1 + m^2]^{-1}$'s and assume that each line has been assigned a fixed value of m.

Next we take care of the monomial in the differences between pairs of external temporal arguments. We have to work a little bit because the propagator for line ℓ has been bounded by $[1 - |t - m_\ell \beta|]^{-N}$ with m_ℓ possibly nonzero. Select a path in G from t_i to t_j. First observe that if any line ℓ on the path has $|m_\ell| \ge 2$ then for all $t = t_{v_\ell, i_\ell} - t_{v'_\ell, i'_\ell}$

$$\frac{1}{1 + |t - m_\ell \beta|} \le \frac{1}{1 + \beta}$$

beats any difference $|t_j - t_{j'}| \le \beta$ between temporal external arguments. So we may as well assume that every $m_\ell \in \{-1, 0, 1\}$. If every line ℓ on the selected path has $m_\ell = 0$ we can apply

$$1 + |t_j - t_{j'}| \le \prod_{i=1}^{a} [1 + |\tau_{i-1} - \tau_i|]$$

where the τ_i are the various temporal arguments along the path. If $t = \tau_{i-1} - \tau_i$ is the time difference between the ends of a propagator, we can cancel it using one factor from the propagator's $\frac{1}{[1+|t|]^N}$. If $t = \tau_{i-1} - \tau_i$ is the time difference between two legs of a common vertex, we will be able to control it using the decay hypothesised in (H4uv). So we may as well assume that at least one propagator on the path has $|m| = 1$. Furthermore, we have chosen the $2q^{\text{th}}$ external leg as one end of the path. That is, we may choose $\tau_0 = 0$. Then $|\tau_{i-1} - \tau_i - m_i \beta| \ge \frac{\beta}{2a}$ for at least one i. (We are defining m_i to be zero when $\tau_{i-1} - \tau_i$ is the time difference between two legs of a common vertex.) To see this, suppose i_0 is the smallest index with $m_i \neq 0$. If $|\tau_{i-1} - \tau_i - m_i\beta| \le \frac{\beta}{2a}$ for every $i < i_0$, we have $|\tau_{i_0-1}| \le (i_0-1)\frac{\beta}{2a}$. As $|\tau_{i_0}| \le \beta/2$ we have $|\tau_{i_0-1} - \tau_{i_0}| \le \frac{\beta}{2} + (i_0-1)\frac{\beta}{2a} \le \beta - \frac{\beta}{2a}$ and hence

$$|\tau_{i_0-1} - \tau_{i_0} - m_{i_0}\beta| \ge \tfrac{\beta}{2a}$$

which implies

$$\frac{1}{|\tau_{i_0-1} - \tau_{i_0} - m_{i_0}\beta|} \le \frac{2a}{\beta}$$

As $2a$ is no more than four times the order of the graph G, we can absorb the $2a$ in the $\mathrm{const}(r)^n$ appearing in the statement of the Proposition.

Finally, we bound the integrals over temporal coordinates, using a tree bound that we now state. Let T be any labelled tree. Denote by $\mathcal{V}$ the set of vertices of T, by $\mathcal{L}$ the set of lines of T and by I_v the incidence number of the vertex $v \in T$. Label the legs of vertex v by $1,\ 2,\ \cdots,\ I_v$. The line $\ell \in T$ joins the $i_\ell^{\rm th}$ leg of vertex v_ℓ to the ${i'_\ell}^{\rm th}$ leg of vertex v'_ℓ. Then, for any functions $F_v : \mathbb{R}^{dI_v} \to \mathbb{C},\ v \in \mathcal{V}$ and $f_\ell : \mathbb{R}^d \to \mathbb{C},\ \ell \in \mathcal{L}$

$$\begin{aligned}\max_{x\in\mathbb{R}^d}\max_{r\in\mathcal{V}}\max_{1\le i_r\le I_r}\int \prod_{v\in\mathcal{V}}|F_v(y_{v,1},\cdots,y_{v,I_v})|\prod_{\ell\in\mathcal{L}}|f_\ell(y_{v_\ell,i_\ell} - y_{v'_\ell,i'_\ell})|\delta\big(y_{r,i_r}-x\big)\prod_{v\in\mathcal{V}}\prod_{i=1}^{I_v} d^d y_{v,i} \\ \le \prod_{\ell\in\mathcal{L}}\|f_\ell\|_1 \prod_{v\in\mathcal{V}}\max_{1\le i\le I_v}\sup_{y_i}\int |F_v(y_1,\cdots,y_{I_v})| \prod_{\substack{1\le j\le I_v \\ j\ne i}} d^d y_j\end{aligned}$$

This is easily proven by induction on the number of lines in the tree.

We select a spanning tree T for G. We bound every $[1+|\tau - m\beta|]^{-N'}$ associated with a line not in T by one. We apply the tree bound with

$$\begin{aligned} f_\ell(y) &= \frac{1}{[1+|y-m_\ell\beta|]^{N_\ell}} \\ F_v(\vec{y}) &= \prod_{i\ne j}[1+|y_i-y_j|]^{r_{ij}}\partial_{\vec{\mathbf{k}}}^{\alpha_v} U_v(\vec{y},\vec{\mathbf{k}}) \end{aligned}$$

The first bound of Proposition III.1 follows by (H4uv) and the fact that

$$\sup_m \int d\tau \frac{1}{1+|\tau - m\beta|^{N'}} \le \mathrm{const}$$

for all $N' \ge 2$.

The proof of the second bound is similar to that of the first. However, we must replace G_β by $G_\beta - G_\infty$ and must extract a factor of β^{-1} as part of the bound. There are two differences between G_β and G_∞. The first is that the propagator for G_∞ is $U(t,\mathbf{k})$ while that for G_β with $\beta < \infty$ is

$$U_\beta(t,\mathbf{k}) = \sum_{m\in\mathbb{Z}} (-1)^m U(t - m\beta, \mathbf{k})$$

The second is that in G_β all temporal components of vertex positions are integrated over $(-\beta/2, \beta/2)$ while in G_∞ they are integrated over $(-\infty,\infty)$. Thus the difference $G_\beta - G_\infty$ consists of all terms from G_β for which at least one line ℓ is assigned $m_\ell \neq 0$ as well as (minus) the portion of G_∞ for which at least one vertex has $|\tau_v| \geq \beta/2$. As well, differentiation with respect to the temporal component of the $j^{\rm th}$ external momentum corresponds, in position space, to multiplication by $(-1)^{j+1} i\tau_j$ for G_∞ and to multiplication by $\frac{\beta}{2\pi}\left(e^{-(-1)^j i\frac{2\pi}{\beta}\tau_j} - 1\right)$ for G_β with $\beta < \infty$.

We have already seen, in discussing the bound on any polynomial in the differences between pairs of external arguments, that as soon as there is one propagator with $m_\ell \neq 0$, we can extract factors of β^{-1} from the temporal decay factors associated with the lines and vertices of G. Similarly, if all propagators have $m_\ell = 0$ and there is at least one vertex with $|t_{v,i}| \geq \beta/2$, as can happen in the evaluation of G_∞, we can extract factors of β^{-1} from a string $\prod_{i=1}^{a} \frac{1}{[1+|\tau_{i-1}-\tau_i - m_i\beta|]^N}$ of temporal decay factors that joins the $2q^{\rm th}$ external vertex to $t_{v,i}$.

Finally, as

$$\left|\frac{\beta}{2\pi}\left(e^{-(-1)^j i\frac{2\pi}{\beta}\tau_j} - 1\right) - (-1)^{j+1} i\tau_j\right| = \frac{\beta}{2\pi}\left|e^x - 1 - x\right|_{x=(-1)^{j+1} i\frac{2\pi}{\beta}\tau_j} \leq \text{const}\,\frac{|\tau_j|^2}{\beta}$$

for all $|\tau_j| \leq \beta/2$, we also get a factor of β^{-1} for the difference between the two definitions of differentiation in temporal directions. ■

§IV References

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