Quantum Electrodynamics in External Fields: Formulation and Consequences of the "Generally Covariant Locality Principle"

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External field QED

- The electromagnetic field separated (uniquely up to a homogeneous solution) into:
  - the classical arena, produced by some external currents and fixed: $A^a(t, \vec{x})$,
  - the quantum (photon) field $\mathcal{A}$.

- The Dirac field ψ propagates on A:

$$(i\gamma^a \partial_a + e\gamma^a A_a - m)\psi = 0,$$

  and causes no back-reaction (test field).

- Interaction of ψ and $\mathcal{A}$ can be included with the help of the causal perturbation theory.

General remarks

- This theory is not fundamental. It is also (probably) much more complicated then a fundamental one ($\sim$ pre-quantum understanding of the H-atom).

- In this way also the QFT in curved spacetimes is constructed. QED is however simpler (trivial causality structure, uniqueness of geodesics etc.)

- It is possible to ask questions about the range of validity of this theory, because the full QED is known. However: substantial (infra-red) problems are present in the QED if there are charges ($?$ universality of the electric charge) ; precisely these charges are meant to produce the external field $A_a(x)$.

Plan of the talk

1. Quantization of the Dirac field in external potentials

2. The locality principle: variations of the external field

3. Consequences of the locality principle

1. Quantization of the free Dirac field (standard)

Task: represent $\psi(f)$, which fulfill the CAR, as operators on a Hilbert space

$$\{\psi(f), \psi^*(g)\} = (f,g)_{\mathcal{H}} = \int d^3x \sum_{B=1}^{4} f_B^+(\vec{x})\, g_B(\vec{x}),$$

here $(f,g)_{\mathcal{H}}$ is the scalar product of test functions (scalar product for classical Dirac fields - elements of $\mathcal{H}$);

$$\psi(f) = \int d^3x\, f^+(\vec{x})_B\, \psi^B(\vec{x}) \qquad f_B(\vec{x}) : \text{bi-spinor test functions}$$

- Given a reference state Ω it is possible to construct a representation of the CAR
- Pure reference states (Ω) are in 1:1 correspondence with projections on $\mathcal{H}$.

Problem: there are plenty of projection operators in $\mathcal{H}$!

Example: ground state representation (static background)

Ground state is defined via a projection on the positive part of the spectrum of the classical Hamiltonian H. The field operator:

$$\psi_B(\vec{x}) = \int d^3p \left[u^s_B(\vec{x},\vec{p})a_s(\vec{p}) + v^s_B(\vec{x},\vec{p})b^*_s(\vec{p})\right] + \sum_n u^s_B(\vec{x},n)a_s(n) + \sum_m v^s_B(\vec{x},m)b^*_s(m$$

$$\psi(f) = a(P_+f) + b^*(P_-f)$$

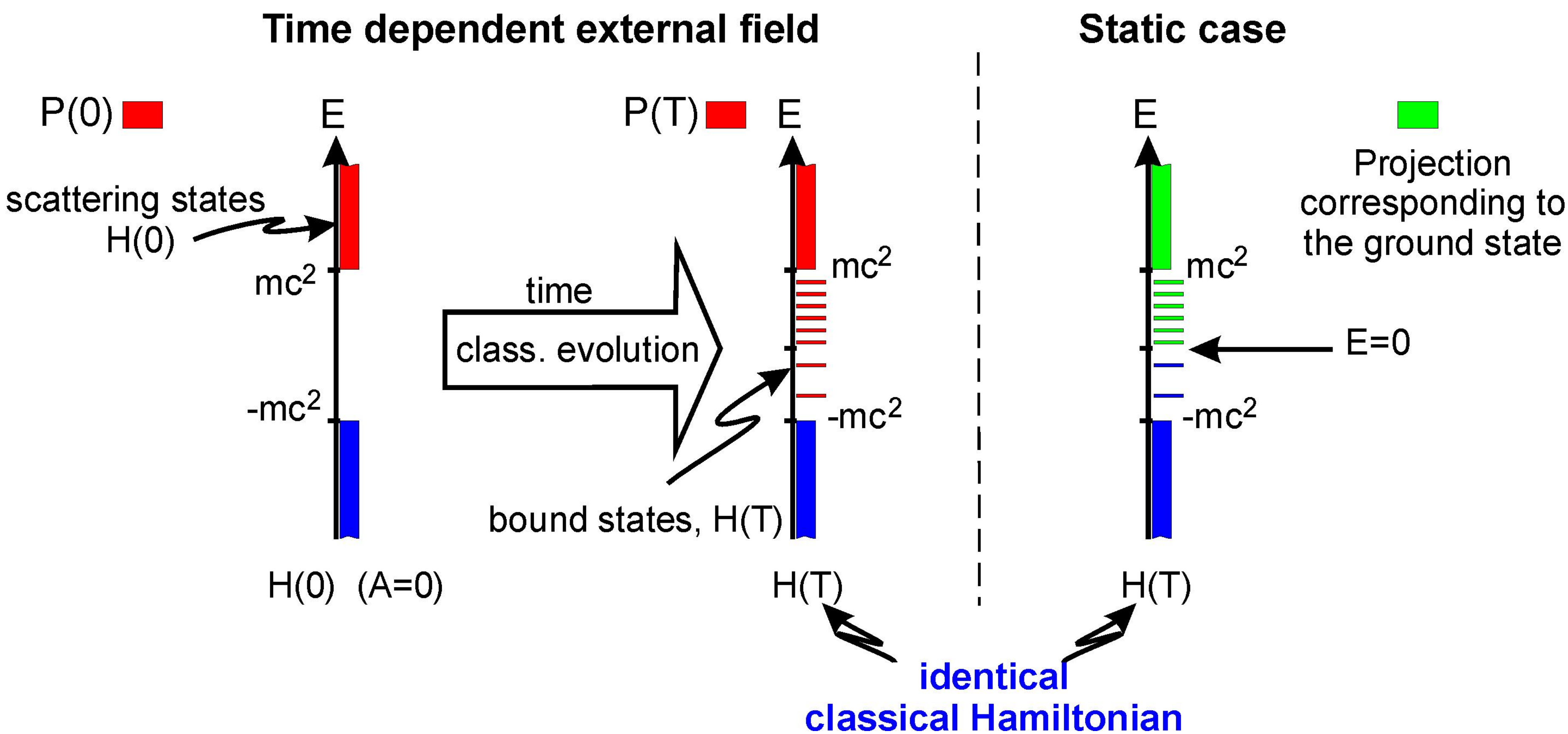
here: $u_B(\vec{x},\vec{p})$ - continuum eigenfunctions (of positive fq-cy); $u_B(\vec{x},n)$ - bound state eigenfunctions.

a's and b's fulfill the standard anti-commutation relations. The representation is based upon the vector Ω:

$$a_\sharp\Omega = 0, \qquad b_\sharp\Omega = 0.$$

Comparison of representations

Example: two physically motivated choices of reference states



States (and the associated representations) Ω, Ω can be: globally equivalent, locally equivalent or disjoint.

- Global equivalence: $\exists$ a unitary $\mathbf{U}$, such that $\Omega = \mathbf{U}\Omega$. There are finitely many excitations in Ω (when measured against Ω).

- Local equivalence: the relative density of excitations in Ω (when measured against Ω) is finite.

- Significant progress is possible if one restricts considerations to the Hadamard states, which all have the same singular part of their two-point functions.
  - If ω is Hadamard in a neighborhood of a Cauchy surface, then it is Hadamard everywhere (for all times)
  - If the external field is static for a definite period of time, then the ground state is a Hadamard state.
  - Any two Hadamard states are locally equivalent.

Non-linear observables and the normal ordering

In QFT all observables must be constructed with the help of field operators. **Nonlinear observables** are much more interesting, then (multi)linear ones (eg. the photodiode-response operator $\int d^3x\ f(\vec{x})\vec{E}(t,\vec{x})\vec{E}(t,\vec{x})$). They involve products of (operator valued) distributions. It is necessary to regularize them

$$j^a(x) = \lim_{y\to x}\big[\overline{\psi}(x)\gamma^a\psi(y) - J^a(x,y)\big].$$

here $J^a(x,y)$ a number-valued distribution. Standard choice:

$$:j^a(x):_\Omega \Longleftrightarrow J^a(x,y) = \big(\Omega, \overline{\psi}(x)\gamma^a\psi(y)\ \Omega\big),$$

is equivalent to the normal-ordering. Problem: which reference state Ω should be employed? Are only differences

$$\big(\Omega, \overline{\psi}(x)\gamma^a\psi(y)\ \Omega\big) - \big(\Omega, \overline{\psi}(x)\gamma^a\psi(y)\ \Omega\big)$$

physically relevant?

2. The locality principle: variations of the external fields

Example from NR quantum mechanics: the current operator

$$j_i(\vec{x}) = \frac{e}{m}\left[-i\hbar\partial_i - \frac{e}{c}A_i(\vec{x})\right].$$

The current operator at $\vec{x}$ depends *evidently* only on $A_i(\vec{x})$. The expectation value of this current corresponds to the electric current of a given state.

In QFT observables are constructed from field operators and distributions. Question: does

$$j_i(x) = \lim_{y\to x}\left[\overline{\psi}(x)\gamma_i\psi(y) - (\Omega, \overline{\psi}(x)\gamma_i\psi(y)\ \Omega)\cdot \mathbf{1}\right]$$

depend locally on the external field?

Formulation of the locality principle for QFT

General formulation: whenever one has two external field arenas, say A and $\tilde{A}$, which for some (arbitrary) open region U are identical (up to a gauge transformation), then it is required that the algebras of observables associated to these regions are isomorphic. These algebras contain non-linear observables (such as currents and interacting-field operators).

Simple consequence: if a distribution t is employed in a definition of an observable, then

$$\frac{\delta\ t(f)}{\delta A(x)} = 0$$

if $x \not\in supp(f)$. (There should, first of all, be a systematic way to define t for different external fields!)

Nonlocality of the two-point function and Hadamard states

For static external fields, the two-point function of the ground state is

$$(\Omega, \overline{\psi}_B(\vec{x})\psi_C(\vec{y})\ \Omega) = \int d^3p \sum_s \overline{v}^s_B(\vec{x},\vec{p}) v^s_C(\vec{y},\vec{p}).$$

The RHS depends *evidently* not only on the external field in a neighborhood of $\vec{x}$ and $\vec{y}$. If expectation values (with respect to some state) are employed in definitions of observables, then these will violate the locality principle (algebras of observables of A and $\tilde{A}$ will not be isomorphic). For Hadamard states, there holds

$$(\Omega, \psi(x)\overline{\psi}(y)\ \Omega) =$$

$$(i\partial\!\!\!/ + eA\!\!\!/ + m)\left(\frac{u(x,y)}{\Gamma} + v_0(x,y)\ln\Gamma + \text{less singular terms}\right) + w(x,y),$$

$\Gamma = (x-y)^2$. The singular term is local (it is the Hadamard parametrix); the smooth term $w(x,y)$ contains *the whole* state-dependent information.

Locality demands, that the algebra of observables supported in a region U remains unchanged if the external field is varied outside of this region. As a consequence no normal-ordering prescription is local.

3. Consequences of the locality principle

- The theory currently employed in the derivation of experimentally testable predictions (such as corrections to the energy levels of highly ionized uranium ions), the Furry picture of QED, appears to be using normal-ordered expressions (see eg. eq. (7) of Phys. Rep. **293** (1998), 227).

- Because these predictions are compared with experiments (the quality of which grows constantly) we ask: in what way the non-locality of normal ordered products affects the predictions?

- A direct calculation of the same predictions from the local QED appears out of scope at the moment. Problem: for realistic (even static) external fields one needs to find (essentially: the expectation value of the local-current density)

$$\lim_{y\to x}\left[(\Omega, \overline{\psi}_B(\vec{x})\psi_C(\vec{y})\ \Omega) - \text{Hadamard parametrix}\right] = \lim_{y\to x} w(x,y).$$

Expectation values of local observables

- Little is known about the local current-densities of ground states.
- A related problem is the expectation value of the local energy-density.
- In there are boundaries (Casimir effect), then energy-densities of ground states generally diverge as one approaches these. The leading divergence is related to the local geometry of the boundary (Candelas).
- For boundaries modeled by smooth external fields the divergence disappears (ground states are Hadamard states).
- So-called quantum inequalities, when applied together with the locality principle, provide general bounds.

Quantum Inequalitites

Let Ω be the ground state for a static arena. QI:

$$\langle \rho(f) \rangle_\omega - \langle \rho(f) \rangle_\Omega \geq -Q_\Omega[f], \tag{1}$$

where

$$\rho(f) = \int dt\ T_{00}(\vec{x}, t)\ f(t), \tag{2}$$

Here: $f(t)$ a probabilistic weight function and $Q_\Omega[f]$, is a functional of f, constructed with the help of the reference state Ω (for instance, $Q_\Omega[f] = \int E^2 |\hat{f}(E)|^2\ dE$).

- The “longer” the support of $f(t)$, the smaller the value of $Q[f]$,
- the energy density of an arbitrary state ω can fall substantially below the ground-state level Ω only for a short period of time.

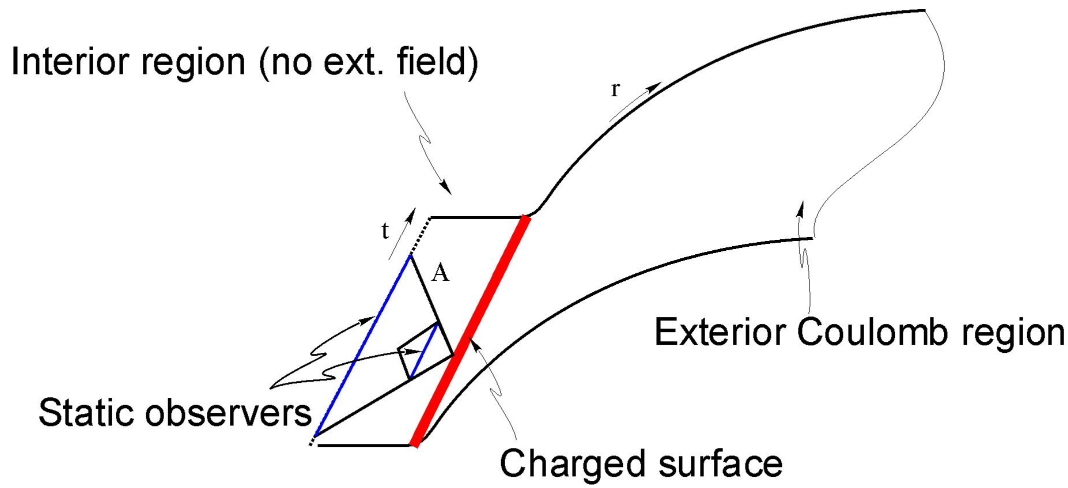


Figure 1: A static arena: in the region A it has no external field. For static observers close to the surface only "small" double-cones can be constructed, which do not include parts of the surface.

Let ω, Ω be two ground states on two spacetimes, with isometric regions. Then we develop QIs with respect to both of these states :

$$Q_{\Omega}(\vec{x})[f] \geq \langle \rho(\vec{x}, f) \rangle_{\Omega} - \langle \rho(\vec{x}, f) \rangle_{\omega} \geq -Q_{\omega}(\vec{x})[f],$$

- The energy-density of the ground state can be approximated by that of the vacuum in free space (which is zero) in the interior region,

- consequently the expectation value of the energy-density of the ground state is small (with a bound provided by QIs) within a region of no external fields; substantial difference is allowed only close to the surface.

- An intuitive understanding can be formed: the difference of expectation values (in ground states) of local and normal-ordered (non-local) observables cannot be substantial within a region of constant external potential.

Summary:

- Quantization of the Dirac field in external potentials suffers from a lack of preferred vacuum state.
- Observables within a region U should depend only on the external field within this region.
- Expectation values of the local energy density differs from the normal-ordered one. The difference is small if there is a region of vanishing external fields.