“E pur si muove” - on the problem(s) of rotation in Gereral Relativity

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Outline of the talk

1. "Gravitomagnetic" effects in General Relativity

2. Homogeneous spacetimes with gravitomagnetic effects

3. Superfluid models of curved spacetimes

Weak field limit of General Relativity:
Perturbations $\bar{h}_{ab}$ of flat spacetime by a spherically symmetric source:

$$ds^2 = [1 + V(r)]\, dt^2 - [1 - V(r)]\, d\vec{x}^2 + A_i\, dx^i\, dt$$

are found from $(\partial_t^2 - \nabla^2)\bar{h}_{ab} = T_{ab}$, which simplify to:

$$\vec{A}(\vec{x}) = \int \frac{\rho \vec{v}(\vec{y})}{|\vec{x} - \vec{y}|}\, d^3y, \qquad V(r) = -\int \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|}\, d^3y.$$

which are *the same as the electromagnetic potentials for the charge conf.* $(\rho, \vec{v})$.

Physical consequences (roughly):
Typical effects associated with $\vec{A}$: as of magnetic fields with $\vec{B} = \mathrm{rot}\vec{A}$, which is the tendency of particles (geodesics) to circulate in the plane perpendicular to $\vec{B}$.

Relativity of inertia in rotating systems

(Newtons bucket and Mach's considerations)

Result for a heavy sphere rotating with angular velocity $\vec{\Omega}$:
inside of the sphere:

- the Newtonian potential $V(\vec{x}) = const$
- the vector potential $\vec{A}(\vec{x}) \sim \vec{\Omega} \times \vec{x}$
- homogeneous $\vec{B}$ - a Coriolis-type effect.

Outside of the sphere:

- the Newtonian potential $V(\vec{x}) = -M/r$
- the vector potential $\vec{A}(\vec{x}) \sim \vec{\Omega} \times \vec{x} \cdot \frac{1}{r^3}$
- $\vec{B}$ drops off as $1/r^3$.

Proportionality factors of order R_{Schw}/R.

Strong-field results

Strong field results for compact objects

- Perturbative method of Hartle: find the spacetime of a rotating star for a given non-rotating configuration. Perturbation in Ω (strong fields).
- Dragging of inertial frames stronger for more compact objects. Usually $\omega_{drag} \ll \Omega$, but if $R \to R_s$ then ω_{drag} becomes a significant fraction of Ω.

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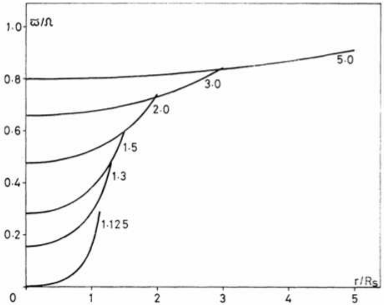


Fig. 1. *The angular velocity $\varpi = (\Omega - \omega)$ relative to the local inertial frame: ϖ/Ω is plotted aginst r/R_S for several values of R/R_S. The curves are labelled by the values of R/R_S to which they belong.*

Figure: Rotating incompressible fluid stars; Chandrasekhar and Miller, MNRAS 167, 63.

Geodesics for (some) rotating spacetimes

$$ds^2 = \left(dt + A_i(\vec{x})dx^i\right)^2 - h_{ij}(\vec{x})\,dx^i\,dx^j, \qquad i, j = 1 \ldots 3$$

- Consider $\vec{A}(\vec{x})$ as a vector field on a surface H (section) with the metric h_{ij}
- Geodesics? Equivalent problem: trajectories in the *static* spacetime

$$ds^2 = (dt)^2 - h_{ij}(\vec{x})\,dx^i\,dx^j$$

  in a magnetic field corresponding to $F_{ij} = \partial_i A_j - \partial_j A_i$.
- More precisely finding geodesics requires solving for the trajectory $\vec{x}(s)$

$$\frac{d^2x^j}{ds^2} + \Gamma^j_{ij}\frac{dx^i}{ds}\frac{dx^k}{ds} = E\,F_{ij}\dot{x}^i$$

  (Γ's of h_{ij}) together with the equation for $t(s)$

$$\dot{t} + A_i\dot{x}^i = E.$$

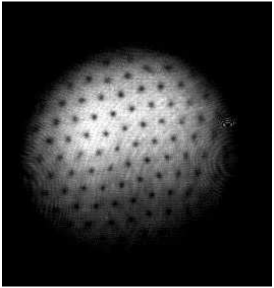
Summary of gravitomagnetic results

Message:

- For weak gravitational fields there is an almost complete analogy between magnetic and gravitomagnetic fields
- Gravitomagnetic fields have an effect similar to a rotating frame of reference; presence of massive rotating bodies alters the definition of an inertial (non-rotating) frame
- Effect much stronger if the “sources” are very compact

Section 2: Homogeneous gravitomagnetic fields

Cold quantum phases rotate/are magnetized in a different way than "normal" phases. Resulting configurations are homogeneous and anisotropic.



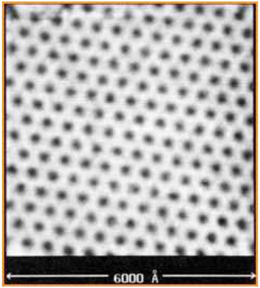


Figure: Lattice of vortices (rotation of a superfluid, MIT Group); Lattice of magnetic-field vortices in type II superconductor (Hess et al. PRL 62, 214).

- quantization of velocity circulation, $\int_{vortex} \vec{v} \cdot \vec{dx} = n\frac{h}{m}$
  quantization of the magnetic flux, $\int_{vortex} \vec{A} \cdot \vec{dx} = n\frac{hc}{2e}$
- cf. velocity profile of a normal rotating fluid
  cf. magnetic field in type I superconductor; frame dragging in Kerr spacetime

Gödel models and their (global) causal structure:

- Gödel's models: the homogeneous section H (h_{ij}) is: Lobachevsky (hyperbolic) plane, sphere or a flat plane plus a free, distinguished direction z.
- gravitomagnetic field of $\vec{A}$, $\vec{B} = (0, 0, B)$, is *homogeneous* on H

$$d(Rs)^2 = \left(dt + H(r)d\varphi\right)^2 - dr^2 - D^2(r)d\varphi^2 - dz^2$$

with

$$H(r) = \begin{cases} 2B\sinh^2(r/2), \\ 2B\sin^2(r/2), \\ \frac{1}{2}\,Br^2, \end{cases} \qquad D(r) = \begin{cases} \sinh(r), \\ \sin(r), \\ r, \end{cases}$$

- parameters: R (scale) and B
- circles $x = const$ are closed *timelike* lines for sufficiently large r; they correspond to (some) "outward" acceleration; by homogeneity - such curves pass through every point.
- projections of light-like and time-like geodesics to (r, φ) are "circles" (special cases of the result for general h_{ij})

Spinless waves in Gödel's spacetime

- Problem: determine solutions of the wave equation

$$\Box\Psi(t,\vec{x}) = \frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}g^{ab}\partial_b\Psi\right] = 0$$

  (usual formula for the Laplacean in curved coordinates; here: spacetime)

- Ansatz: general solution Ψ is a linear combination of solutions determined by separation of variables,

$$\Psi(t,\vec{x}) = \sum_{I=(E,p,\ldots)} c_I\Psi_I(t,\vec{x}), \qquad \Psi_I(t,\vec{x}) = e^{-iEt}e^{ipz}\psi(r,\varphi)$$

  (a sum, not insisting on "initial value formulation")

- In all cases, all solutions can be found explicitly. In most cases: by algebraic methods (ladder operators). Resulting functions are elementary functions of $w = \tanh(x/2)e^{i\varphi}$, e.g. $\psi_{LLL} = (1-w\overline{w})^\lambda\overline{w}^m$ with $\lambda\in\mathbb{R}_+$, $m\in\mathbb{N}$.

Sketch of the solution...

- There are five Killing vectors (generators of symmetries). Three of them K_0, K_1, K_2 fulfill the SU(1,1) algebra commutation relations. The remaining ones are $K_T^a = (\partial_t)^a$ and $K_z^a = (\partial_z)^a$.
- Remarkable identity:

$$\Box = \underbrace{(K_1^2 + K_2^2 - K_0^2)}_{\text{Casimir op. of } SU(1,1)} + \underbrace{(1 - B^2)(\partial_t)^2 - (\partial_z)^2}_{\text{lin. comb. of } K_T^2 \text{ and } K_z^2}$$

- In Gödels original case: $B = \sqrt{2}$; using our Ansatz it remains to determine eigenvectors of the Casimir operator to *positive* eigenvalues.
- This can be done algebraically (as for spherical harmonics), note that

$$K_+ = \quad w^2\partial - \overline{\partial} - (EB)w,$$
$$K_- = -\overline{w}^2\overline{\partial} + \partial - (EB)\overline{w},$$
$$K_0 = w\partial - \overline{w}\overline{\partial} - (EB), \qquad \partial = \frac{\partial}{\partial w}$$

- We get base vectors annihilated by K_- (and K_+), and generate the rest by applying K_+ (or K_-).

Summary the results for homogeneous spacetimes:

Message:

- Gödel's spacetimes are gravitational analogs of constant magnetic fields.
- Geodesics (trajectories of classical particles) and wave equations can be solved in these spacetimes exactly and are expressed by elementary functions.

Observation made by W. Unruh:

Unruh extended the usual derivation of wave equation in hydrodynamics to arbitrary (irrotational) flows $\rho(t,\vec{x})$, $\vec{v}(t,\vec{x})$, and small perturbations of this flow $(\delta\rho,\delta\vec{v})$,

- parametrize the perturbations via $\delta\vec{v} = \vec{\nabla}\phi, \quad \delta\vec{\rho} = -\frac{\rho}{c^2}\,\frac{d}{dt}\,\phi$ with $(\partial_t + \vec{v}\cdot\vec{\nabla})$, $c^2 = \frac{\partial p}{\partial \rho}$
- equations of hydrodynamics, fulfilled by $(\rho,\vec{v})$, lead to an equation for ϕ:

$$\frac{d}{dt}\left[\frac{1}{c^2}\,\frac{d\phi}{dt}\right] - \frac{1}{\rho}\,\partial_i(\rho\,\partial^i\psi) = 0.$$

- this is exactly the wave equation for spin-0 fields on the spacetime

$$ds^2 = \frac{\rho}{c}\,\left[c^2\,dt^2 - (\vec{dx} - \vec{v}\,dt)^2\right]$$

Rotating irrotational flows and their acoustic spacetimes

- Analogy works only for irrotational background flows, $\nabla \times \vec{v} = 0$
- There are so-called irrotational vortices in hydrodynamics; $\vec{v} \parallel \hat{\phi}$, behaving as $|\vec{v}| = \frac{const}{r}$
- Often: finite empty core (always $\rho(0) = 0$); for normal fluids - models break down within a finite core, which often rotates rigidly (analogy does not work)
- Superfluids fulfill the assumptions perfectly... for distances larger than the (small) healing length, where the "quantum pressures" modify the hydrodynamic equations.
- Stationary, axially symmetric spacetimes without the axis of rotation (if core empty) are pathological (way too ambiguous); one cannot exclude redefinitions of (t, φ) variables leading to scaling of the angle and introductions of regions with CTCs (gravitational Abrikosov vortices)
- Comparison with superfluid sound-scattering data provides a tool for determining an effective modification of the rotating spacetime in the core region

Two questions...

1. What is the acoustic spacetime corresponding to a (very large) lattice of vortices?

2. Is it possible to find an acoustic model of locally rotating spacetimes of Gödel type (which share the symmetry of the vortex lattices)?

Summary of the talk

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- Rotating matter in GR leads to the appearance of "gravitomagnetic" fields, which relativize (set the context for) our notion of non-rotating frames.
- Among examples of this effect the simplest are provided by Gödels spacetimes, which are *homogeneous* and anisotropic
- Physics in Gödel's spacetimes is not technically too difficult, but can be conceptually demanding...
- Small sound on given background superfluid flows provides an arena essentially equivalent to test spin-0 fields in curved spacetimes. To some extent rotating spacetimes can be modeled in this way, but challenges remain in the core of vortices.
- It is not clear what type of a spacetime corresponds to a rotating vortex-lattice.