External field Quantum Electrodynamics from the local point of view

Piotr Marecki
II Institut für theoretische Physik
Universität Hamburg
marecki@mail.desy.de

February 6, 2003

Plan

The external field problem for QED is formulated as follows: the electromagnetic field is divided into two parts, the classical one $A^{\mu}(x)$ given and fixed once and for all, and the quantum one $A_q^{\mu}(x)$ which is the photon field.

The motivation for such a theory comes from the fact, that it shares many difficulties with the QED on a curved spacetime. Construction of external field QED together with its experimental verification would further justify the QED on CST.

The plan of the talk is the following:

1. External field QED (the basics)

2. Local, causal, perturbative QED

3. Experimental verification

External field QED

In order to construct the free external field QED one starts from the classical Dirac field in the external potential $A^\mu(x)$. The time evolution is than generated by the Hamiltonian:

$$H = \gamma^0\gamma^i[-i\partial_i + eA_i(t,\mathbf{x})] - eA_0(t,\mathbf{x}) + m\gamma^0.$$

Suppose the external field is time independent.

The spectrum of the Dirac operator than contains a continuous part (scattering states) usually for $E \in (-\infty, -mc^2) \cup (mc^2, \infty)$ and a ladder of bound states.

The theory can be quantized algebraically by means of the CAR

$$\{\psi(x), \psi^*(y)\} = S(x,y),$$

where $S = S_{ret} - S_{av}$ is the anti-commutator distribution which is the unique homogeneous solution with support in the light cone.

However in the time-independent case it is relatively easy to construct the vacuum representation which also defines the algebra. One divides the spectrum into positive and negative frequency parts and defines the quantized field via:

$$\psi_A(x) = \int d^3p \, [u_A(x,p)a(p) + v_A(x,p)b^*(p)] + \\ + \sum_n u_A(x,n)a(n) + \sum_m v_A(x,m)b^*(m) \quad (1)$$

together with CAR

$$\{a(p), a^*(k)\} = \delta(p-k) \quad \{b(p), b^*(k)\} = \delta(p-k)$$
$$\{a(n), a^*(m)\} = \delta_{nm} \quad \{b(n), b^*(m)\} = \delta_{nm}$$
$$\{a_\cdot, b_\cdot\} = 0 \quad \{a_\cdot, b^*_\cdot\} = 0$$

The choice of the positive frequency part is conveniently described by a projection operator on the 1-particle Hilbert space. This projection also fixes

the vacuum representation by

$$a(\mathbf{p})\Omega = 0 \qquad b(\mathbf{p})\Omega = 0.$$
$$a_n\Omega = 0 \qquad b_n\Omega = 0.$$

In the theory quantized in such a way there are two two-point functions, namely

$$iG_{+AB}(x,y) = (\Omega, \psi_A(x)\overline{\psi}_B(y)\Omega)$$
$$iG_{-AB}(x,y) = (\Omega, \overline{\psi}_A(x)\psi_B(y)\Omega)$$

which depend on the choice of the projection. They are constrained by

$$G_+(x,y) + G_-(y,x) = S(x,y)$$

i.e. they sum up to the representation-independent anti-commutator distribution.

Nonlinear fields are of great interest, above all the free electronic current operator, which is classically given by

$$j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x).$$

In QFT such an operator must be defined by a substraction of certain number valued distribution $J^\mu(x,y)$ from the operator valued distribution

$$j^\mu(x,y) = \overline{\psi}(x)\gamma^\mu\psi(y).$$

and a subsequent limit $x \to y$. In standard theory the choice is typically (normal ordering)

$$J^\mu(x,y) = (\Omega, j^\mu(x,y)\Omega).$$

Specifically the current density is modified by

$$J^0(x,y) = i\gamma^0 G_-(x,y).$$

The external field QED is the interacting theory based upon the "distinguished", vacuum representation of the free field algebra. Quadratic observables are defined by a suitable subtraction of the appropriate two-point functions.

Local, causal, perturbative QED

The S-matrix is by definition given the Dyson series:

$$\mathsf{S} = 1 + \int d^4x \; g(x)\mathcal{L}_I(x) + \\ + \int d^4x \; d^4y \; g(x)g(y)T_2(x,y) + \ldots$$

where $T_2(x,y)$ is the second order time-ordered product (Operator Valued Distribution) constructed from the interaction Lagrangean $\mathcal{L}_I(x)$. The g's are test functions which are equal to 1 in the region where the electromagnetic interaction is supposed to take place.

The postulate of locality

> All elements of the algebra (linear and non-linear OVD smeared with test functions) are allowed to depend only on the external field present in the causal neighborhood containing the support of the test functions.

More specifically we construct the time ordered products (TOP) $T_n(x_1 \dots x_n)$ which through the Dyson-series determine the evolution of the observables. Those TOP's with help of the Wick expansion can be shown to consist of the Wick products multiplied with number valued distributions. We require both of them to be local. In particular the Wick product of two field operators

$$: \psi(x)\overline{\psi}(y) := \psi(x)\overline{\psi}(y) - H(x,y)$$

is local if the regularising distribution, $H(x,y)$ fulfills

$$\frac{\delta H(x,y)}{\delta A^{\mu}(z)} = 0$$

if z does not belong to the causal neighborhood containing x and y.

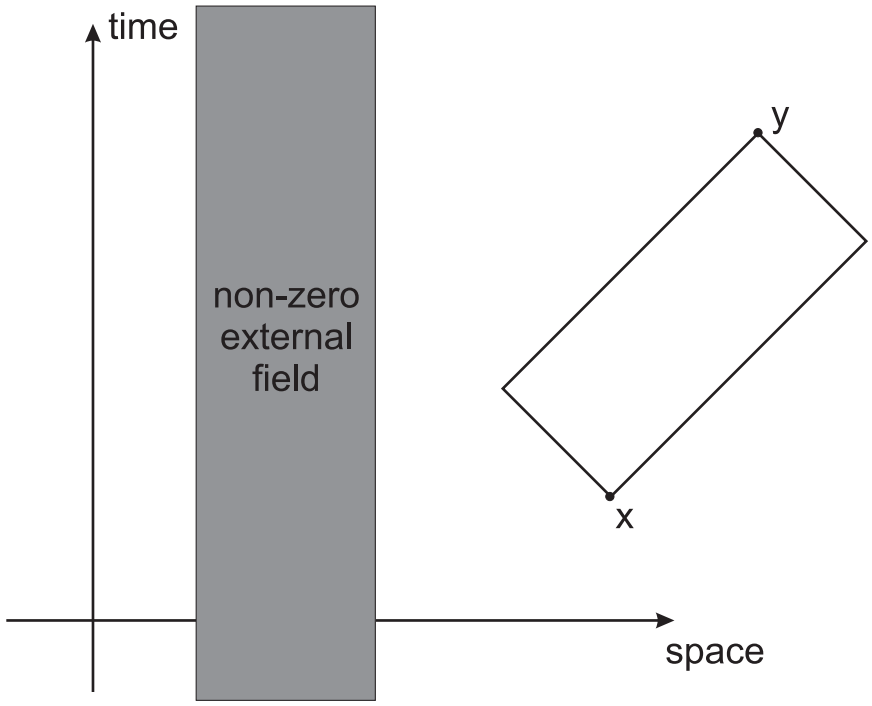

Figure 1: The two-argument (x, y) o.v.d.'s are allowed to depend on the external field only in the region $J^+(x) \cap J^-(y)$ (which is the smallest causal neighborhood containing x and y) and thus - should not feel any effect of the external field.

The two-point functions $G_{\pm}(x, y)$ are not local

Argument: Consider for simplicity the massless, hermitean, scalar field. Let the external potential be only x-dependent. Investigate the scattering, stationary wavefunctions in order to define the field operator with their help.

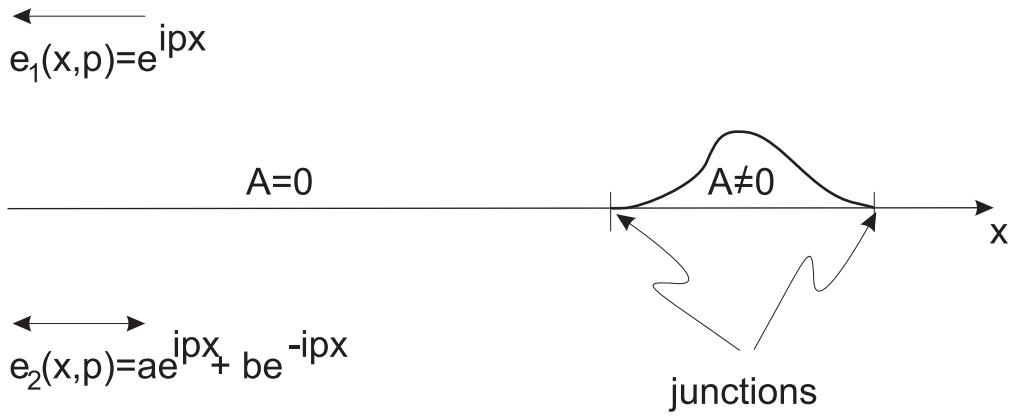

Figure 2: Two independent solutions $e_1(x)$ and $e_2(x)$. The other dimensions (t, y, z) are suppressed.

The y and z dependence factorizes:

$$u(t, x, x_\perp) = e^{iE(p_\perp, p)t - ip_\perp x_\perp} \cdot e(p, x)$$

As there is no external field for small x-es both independent solutions $e_{1,2}(x,p)$ will be linear combinations of the running waves: outgoing e^{-ipx} and incoming e^{ipx}. If we take as the first solution the one for which $e_1 = e^{-ipx}$ for small x than the second one will have the form

$$e_2 = ae^{-ipx} + be^{ipx},$$

where $a \neq 0 \neq b$. Now the field can be quantized:

$$\psi(t,x,x_\perp) = \int_{\mathsf{R}^2} d^2p_\perp \int_0^\infty \frac{dp}{\sqrt{2E}} \cdot$$
$$\left\{e^{iEt-ip_\perp x_\perp}[e_1(x,p)a_1(\mathbf{p}) + e_2(x,p)a_2(\mathbf{p})] + H.c.\right\}$$

with CCR

$$[a_1(\mathbf{p}), a_1^*(\mathbf{k})] = \delta(\mathbf{p}-\mathbf{k}) \qquad [a_1(\mathbf{p}), a_2^*(\mathbf{k})] = 0$$
$$[a_2(\mathbf{p}), a_2^*(\mathbf{k})] = \delta(\mathbf{p}-\mathbf{k}) \qquad [a_.(\mathbf{p}), a_.(\mathbf{k})] = 0$$

The vacuum representation is defined by requiring

$$a_{\cdot}(\mathbf{p})\Omega = 0$$

Now the two point function $(\Omega, \psi(\mathbf{x})\psi(\mathbf{y})\ \Omega)$ (for equal times) is given by

$$\int_{\mathsf{R}^2} d^2p_\perp \int_0^\infty \frac{dp}{2E}\ \cdot \\ \left\{ e^{-ip_\perp(x-y)_\perp}[e_1(x)\overline{e}_1(y) + e_2(x)\overline{e}_2(y)] \right\} \quad (2)$$

The square bracket is evaluated to be

$$[\ldots] = (1+|a|^2)e^{-ip(x-y)} + |b|^2 e^{ip(x-y)} + \\ a\overline{b}\ e^{ip(x+y)} + \overline{a}b\ e^{-ip(x+y)}. \quad (3)$$

On the other hand in the absence of the external field we have

$$[\ldots]_{Minkowski} = e^{-ip(x-y)} + e^{ip(x-y)}.$$

Both distributions are easily seen not to be equal. □

Local definition of the Wick product

The goal is to find a suitable distributions $H, \tilde{H}$ such that they are local and make the limit $x \to y$ of

$$\psi_A(x)\overline{\psi}_B(y) - H_{AB}(x,y)$$
$$\psi_A(x)\overline{\psi}_B(y) - \tilde{H}_{AB}(x,y)$$

well defined as operator valued distributions. The latter condition is fulfilled if and only if the limit $x \to y$ of

$$(\Omega, \psi_A(x)\overline{\psi}_B(y)\Omega) - H_{AB}(x,y)$$
$$(\Omega, \psi_A(x)\overline{\psi}_B(y)\Omega) - \tilde{H}_{AB}(x,y)$$

exists as a distribution.

Both above two-point functions have singularities on the light-cone $\Gamma = (x-y)^2$ and are solutions of the Dirac equation. The method of Hadamard allows us to find distributions with the same properties which are additionally local.

The outline of the Hadamard method

- the sought distribution should fulfill

$$\{i\gamma^a[\partial_a - ieA_a(x)] - m\}H(x,z) \equiv D_x H(x,z) = 0$$

- it is found from

$$H(x,z) = \hat{D}_x \phi(x,z)$$
$$\hat{D}_x \equiv \{i\gamma^a[\partial_a - ieA_a(x)] + m\},$$

where now $\phi(x,z)$ solves the equation

$$D_x \hat{D}_x \phi(x,z) = 0,$$
$$D_x \hat{D}_x \cong [\Box + ieA^a(x)\partial_a + c(x)]$$
$$- e[\gamma^a, \gamma^b]\ F_{ab}(x)$$

- if $\phi(x,z)$ has a singularity at the light cone, than as

a homogeneous solution it must be of the form

$$\phi(x,z) = \frac{u(x,z)}{\Gamma_\epsilon} + \sum_{n=0}^{\infty} v_n(x,z) \ln \Gamma_\epsilon,$$

(modulo smooth function),

- the smooth coefficients u, v_n are determined from the recursive system of ordinary differential equations:

$$\frac{du}{ds} + u \cdot [T_a \; iA^a(y)] = 0$$

$$s\frac{dv_0}{ds} + v_0[1 + sT_a \; iA^a(y)] = -\mathcal{D}_y u$$

$$s\frac{dv_n}{ds} + v_n\,[1 + sT_a \; iA^a(y)] = -\frac{1}{n}\mathcal{D}_y v_{n-1},$$

for which $u(z,z)$ and continuity of $v_n(x,z)$ are the only boundary conditions. Those equations are integrated along the straight line emerging from z. T_a is the (unit) tangent vector; s is the distance, $y = y(s) = z + sT$.

Not a difficult integration leads to the following results:

$$u(x,z) = \exp\left[ie\int_0^s T_a A^a(z+\tau T)\ d\tau\right]$$

$$v_0(x,z) = -\frac{u(z,x)}{4s}\int_0^s \frac{\mathcal{D}_y u(y,z)}{u(y,z)}\ d\tau.$$

The above solutions are smooth and transparently local. The same is true for all v_n's, therefore the distribution $H(x,z)$ is a local one. It possesses all the singularity structure of the two-point function (at least if the latter does not posses any other singularities) and as such it can be used in the definition of the Wick product. However a smooth function can always be added to $H(x,y)$ without changing its properties.

Causal, perturbative QED attempts to construct TOP and other non-linear OVD's. If they are to depend locally on the external field than the two-point functions should be avoided in their definition. Hadamard method provides transparently local regularizing distributions.

Experimental verification

The study of the experimental consequences of the external field QED requires extraordinary measures. On the one hand we would like the external field to be "strong" so as to produce non-trivial effects, on the other it is desirable to avoid all poorly understood phenomena which might appear in connection with the strong fields. Moreover the field should be such that the external field approximation is justified.

In our opinion any experimental test of the external field QED should fulfill the following requirements:

1. The external field should be appropriate, i.e. it should:
   - be well characterized,
   - be classical and suffer little from the back-reaction.
2. The experiment should be clean from any other unknown or poorly characterized interactions.

One experimental area which appears to fulfill all the requirements seems to be the spectroscopy of fully ionized, heavy ions. In such experiments a bare ion is allowed to capture an electron. The energy levels of such an electron are sensitive to the quantum-electrodynamic effects.

The system of bare uranium ion with the $1S$ binding energy of 130keV is characterized as follows:

- the external field outside of the nucleus is the Coulomb field which is classical and fixed by the Gauss law,
- there is virtually no back-reaction on the external field
- the only Pandora's box, the nucleus, (still) has a negligible effect ~ 0.1eV
- no electron-electron interactions obscure the results

Short overview of the experiment

- accelerated uranium ions are stripped from all electrons upon hitting a thin, metal foil target,
- the bare ions are then accumulated in a storage ring (up to 10^8 ions with long. velocity spread of 50ppm),
- the ions hit a gas target where they capture electrons,
- the electrons drop through the energy levels, the x-rays of those transitions are detected,
- the Lyman $2p_{3/2} \rightarrow 1s$ line reveals the $1s$ Lamb shift
- coincidence arrangement assures no more than one electron/ion contributes to the spectra

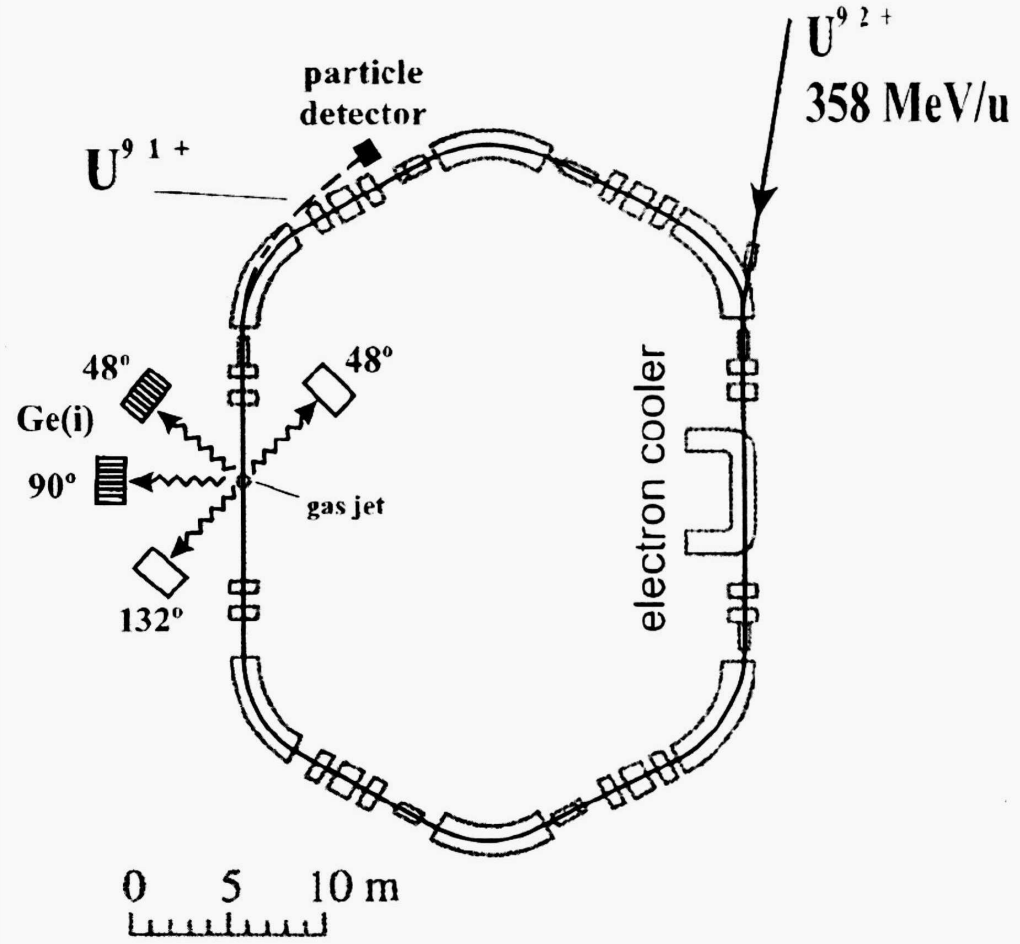

FIG. 1. Schematic diagram of the experimental arrangement at the ESR storage ring. Projectile x rays are registered by four Ge(i) detectors viewing the gas-jet interaction region at angles of 48°, 90°, and 132°. The down-charged U^{91+} ions are separated by a dipole magnet and detected by a fast plastic scintillator counter.

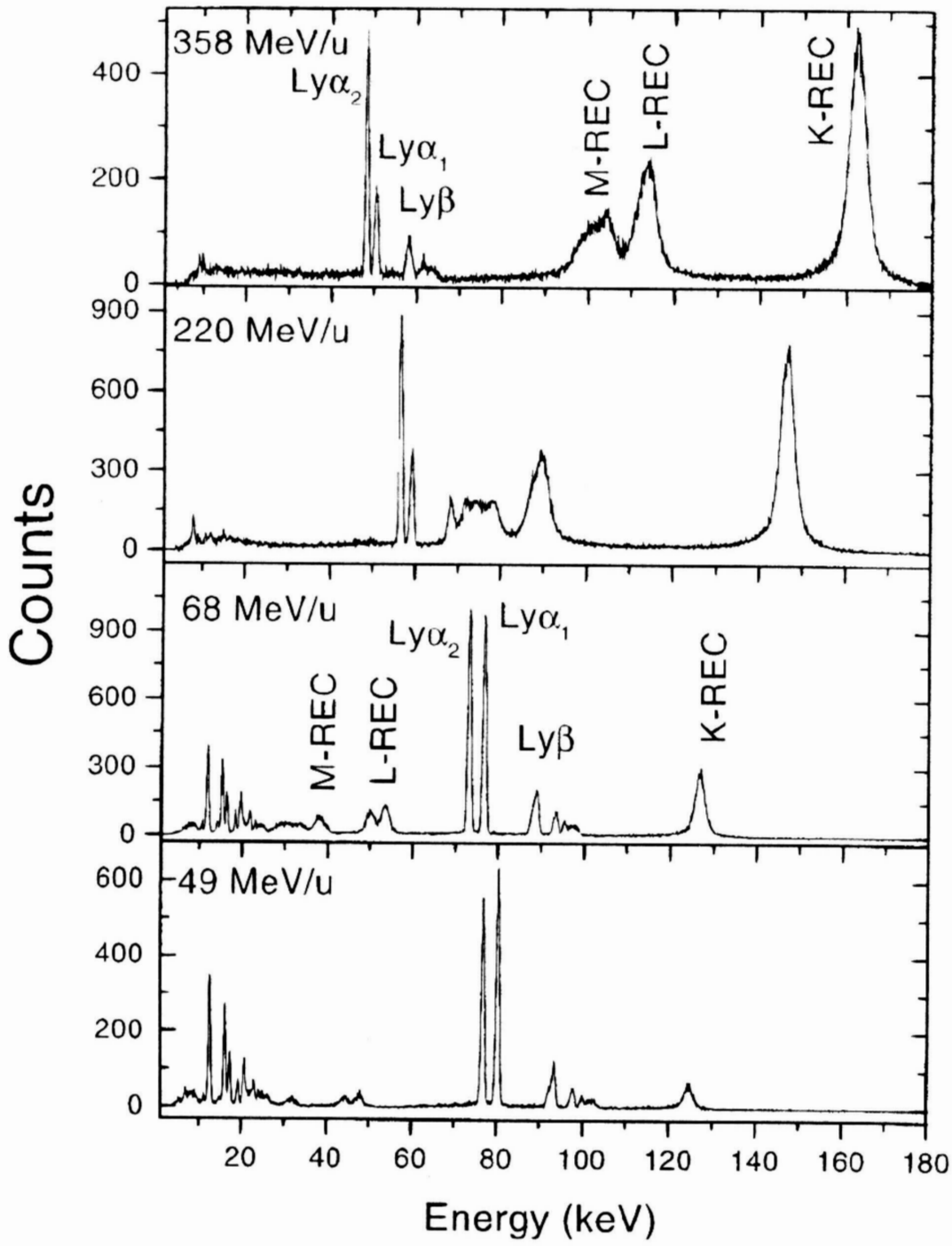

FIG. 2. Typical x-ray spectra registered in coincidence with the particle counter for 49, 68, 220, and 358 MeV/u U^{92+} interacting with N_2 gas targets. The spectra (laboratory frame) were measured at the observation angle of 132° and are not corrected for detection efficiency.

Conclusions and outlook

The following points are important:

- standard theory is based upon vacuum representation and normal ordered expressions
- local theory makes some important adjustments the consequences of which are yet to be determined
- well-prepared experimental setups make the external field regime experimentally accessible