Solutions to the d'Alembert equation in Gödel spacetimes

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Introduction to the problems addressed in this talk

What are spacetimes of Gödel type?

- "Third" in the hierarchy of simplicity after Minkowski and highly symmetric de Sitter/Einstein static Universe.
- Stationary, homogeneous, axisymmetric spacetimes (Lorentzian 3+1 manifolds).
- Non globally-hyperbolic, with closed time-like curves
- Original Gödel spacetime: the source (T_{ab}) is dust + cosmological constant
- Three versions: constructed on 2D homogeneous surfaces with spherical/hyperbolic/flat geometry. On these surfaces there is a "gravitomagnetic field". The "strength" of this filed and the curvature radius lead to a 2-parameter family. The spacetimes are *anisotropic*.

Outline of the talk

1. Motivation: “magnetic” effects in GR
2. Geodesics
3. Waves (d’Alembert equation)

Section 1: Gravitomagnetism/rotating spacetimes

Weak field limit of GR

Asymptotically (far away):

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 + \frac{2m}{r})d\vec{x}^2 + A_i\, dx^i\, dt$$

with m:mass, and $A_i\, dx^i = \frac{J}{r}\sin(\theta)\, d\varphi$ gravitomagnetic effect due to the angular momentum J.
Generally for stationary fluid flows:

$$\vec{A}(\vec{x}) = \int \frac{\rho\vec{v}}{|\vec{x} - \vec{y}|}\, d^3y$$

which is *the same as the electromagnetic potential of a rotating charge distribution.*

Physical consequences

Typical effects associated with $\vec{A}$: as of magnetic fields with $\vec{B} = \mathrm{rot}\vec{A}$, which is the tendency of particles (geodesics) to circulate in the plane perpendicular to $\vec{B}$.
Result for a sphere: homogeneous $\vec{B}$ inside - relativity of centrifugal effect.

Strong-field results; dragging of inertial frames.

Kerr spacetime

In exact solutions the effect is visible as "dragging of inertial frames", i.e. non-zero angular velocity (w.r.t. stars at ∞) by zero angular-momentum observers.

Strong field results for compact objects

- Perturbative method of Hartle: find the spacetime of a rotating star for a given non-rotating configuration. Perturbation in Ω (strong fields).
- Dragging of inertial frames becomes stronger if the object becomes more compact. Usually $\omega_{drag} \ll \Omega$, but if $R \to R_s$ than ω_{drag} becomes a significant fraction of Ω (but this is difficult for stars).
- Description of strongly rotating compact objects in GR is a notoriously difficult problem.

Motivation for Gödel's spacetimes

Are Gödel's spacetimes physically relevant?

- Gödel's spacetimes provide solutions with *purely "magnetic" effects*
- The full spacetime is not astrophysically relevant.
- Fascinating astrophysical phenomena associated with rotating compact objects; Kerr (outer) horizon behaves a little like a superconductor (Bicak)
- Deep physics of cold rotating phases, or conducting phases in magnetic fields

What can be addressed in Gödel's spacetimes?

- Simplest examples of dragging of inertial frames and of gravimagnetism
- Physics (mechanics, classical and quantum fields) in curved spacetimes.
- Classical fields: explicit exact results (this talk).
- Quantum fields: staggering arena (no global hyperbolicity), but: so far only QFT has an argument against causal pathologies (Kay-Radzikowski-Wald)

Summary of motivation

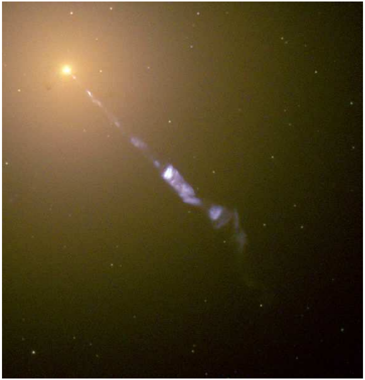

Figure: Relativistic jet from M87 (image: HST).

J.P.Lasota writes in astro-ph/0607453: *"The jet launching mechanism is unknown. This is rather embarrassing and some well-intentioned authors prefer to write that it is the details of this mechanism that are unknown, but this is a rather huge understatement. In most models of jet launching the accretion-flow anchored magnetic field plays a crucial role."*

Consider the (stationary) Lorentzian geometries given by the metrics

$$ds^2 = \left(dt + A_i(\vec{x})dx^i\right)^2 - h_{ij}(\vec{x})\, dx^i\, dx^j$$

with $i, j = 1 \dots 3$.

- The vector field $\vec{A}(\vec{x})$ can be as a vector field on the Riemanninan surface H (section) with the metric h_{ij}
- Geodesics? Equivalent problem: trajectories in the *static* spacetime

  $$ds^2 = (dt)^2 - h_{ij}(\vec{x})\, dx^i\, dx^j$$

  in a magnetic field corresponding to $F_{ij} = \partial_i A_j - \partial_j A_i$. More precisely finding geodesics requires solving for the trajectory $\vec{x}(s)$

  $$\dot{x}^i \nabla_i^{(h)} \dot{x}_j = E\, F_{ij} \dot{x}^i$$

  together with the equation for $t(s)$

  $$\dot{t} + A_i \dot{x}^i = E.$$

  Energy, $E = (\partial_t)^a \dot{x}^b\, g_{ab} > 0$ for for future-oriented lines.

Gödel models and their (global) causal structure:

- Gödel's models: the homogeneous section H (h_{ij}) is $\mathbb{R}\times$ two-sphere $\mathbb{S}_2$, *or* Lobachevsky (hyperbolic) plane $\mathbb{H}_2$ *or* a flat plane $\mathbb{R}^2$.
- the magnetic-field of $\vec{A}$, $\vec{B} = (0, 0, B)$, is *homogeneous* on H; distinguished direction will be called X. Flat case:

$$ds^2 = (dt + \tfrac{1}{2}Br^2 d\varphi)^2 - dr^2 - r^2 d\varphi^2 - dX^2$$

  Spherical case:

$$ds^2 = \left[dt + 2B\sin^2(\tfrac{\theta}{2})d\varphi\right]^2 - d\theta^2 - \sin^2(\theta)d\varphi^2 - dX^2$$

- axial symmetry apparent; $\varphi \in [0, 2\pi)$ with periodicity assumed
- integral curves of $(\partial_\varphi)^a$ are closed timelike lines for $B^2r^2 > 4$; they correspond to (some) "outward" acceleration; by homogeneity - such curves pass thru every point.
- projections of light-like and time-like geodesics to (r, φ) are "circles"

Geometry of simple "rotating" spacetimes:

In the class of geometries with metrics $ds^2 = (dt + \vec{A} \cdot d\vec{x})^2 - h_{ij} dx^i dx^j$ much can be understood. Finding geodesics is equivalent to finding trajectories of charged particles on H in magnetic field $\vec{B} = \mathrm{rot}\, \vec{A}$.

Gödel spacetimes are simplest realizations of this structure with homogeneous (flat, spherical and hyperbolic) H's and a constant unidirectional magnetic field.

Causality is violated in these spacetimes (thus: they are not globally hyperbolic).

Section 3: The d'Alembert equation

- Problem: determine solutions of the linear PDE (d'Alembert)

$$\nabla_a \nabla^a \Psi(t, \vec{x}) = 0$$

- Ansatz: general solution Ψ is a linear combination of solutions determined by separation of variables,

$$\Psi(t, \vec{x}) = \sum_{I=(\omega,P,\ldots)} c_I \Psi_I(t, \vec{x}), \qquad \Psi_I(t, \vec{x}) = e^{-i\omega t} e^{iPX} \psi(r, \varphi)$$

- There are five Killing vectors (generators of symmetries) in spherical (or hyperbolic) cases. Three of them K_0, K_1, K_2 fulfill the SU(2) (or SU(1,1)) algebra commutation relations. The remaining ones are $K_T^a = (\partial_t)^a$ and $K_X^a = (\partial_X)^a$.
- Remarkable identities

$$\nabla_a \nabla^a_{\mathbb{H}_2} = \underbrace{(K_1^2 + K_2^2 - K_0^2)}_{\text{Casimir op. of } SU(1,1)} + \underbrace{(1 - B^2)(\partial_t)^2 - (\partial_X)^2}_{\text{lin. comb. of } K_T^2 \text{ and } K_X^2}$$

$$\nabla_a \nabla^a_{\mathbb{S}_2} = \underbrace{(K_1^2 + K_2^2 + K_0^2)}_{\text{Casimir op. of } SU(2)} + \underbrace{(1 + B^2)(\partial_t)^2 - (\partial_X)^2}_{\text{lin. comb. of } K_T^2 \text{ and } K_X^2}$$

Spherical case: algebraic methods

We take $\Psi = e^{-i\omega t} e^{iPX} \psi(\theta, \varphi)$. The Killing vectors are of the form

$$K_+ = L_+ + B\, z \cdot (i\partial_t)$$
$$K_- = L_- + B\, z \cdot (i\partial_t)$$
$$K_0 = L_0 + B\ \cdot (i\partial_t)$$

with $K_\pm = K_1 \pm iK_2$, and provide a modification of generators $(\vec{L})$ of rotation on the sphere. Here $z = \tan(\frac{\theta}{2}) e^{i\varphi}$. The $\vec{K}$'s are selfadjoint on the Hilbert space of L^2 functions of the sphere with the standard measure.

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- Ladder-operator construction of eigenvectors of $\vec{K}^2$ is standard, provided there exist lowest vectors annihilated by K_- (otherwise$\rightarrow$ singular solutions)
- Eigenvalues of $\vec{K}^2$: $\lambda(\lambda + 1)$ with $\lambda = \frac{\mathbb{N}}{2}$
- for each λ there is a family of vectors to $K_0 = -\lambda, \ldots, \lambda$ (ladder)

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- for each λ there is a family of vectors to $K_0 = -\lambda, \ldots, \lambda$ (ladder)
- for the extremal vector(s) from $K_- \psi_{-\lambda} = 0$ we obtain

$$\psi_{-\lambda} = \cos^{2\lambda - m}\theta \, \sin^m \theta e^{im\varphi}, \qquad \text{where } m = \lambda + B\omega$$

- from periodicity follows $m \in \mathbb{Z}$, solutions singular unless $m \in [0, 2\lambda]$

The resulting structure,

$$\Psi_{\omega,\lambda,k,P}(t,\vec{x}) = e^{-i\omega t} e^{iPX}\ {}_{(B\omega)}Y_{\lambda k}(\theta,\varphi)$$

- Solutions exist for $B\omega = \mathbb{Z}/2$
- For each such frequency there exist families with $\lambda \geqslant |B\omega|$
- In each such family there are eigenfunctions ${}_{(B\omega)}Y_{\lambda k}(\theta,\varphi)$ for $k \in [-\lambda, \lambda]$. They are Spin-($B\omega$) spherical harmonics
- The momentum in the inhomogeneous direction is *discrete*; the wave equation:
$$P^2 = \left(1 + B^{-2}\right)(B\omega)^2 - \lambda(\lambda+1)$$
- This puts an upper constraint on λ for each $(B\omega)$, and produces a gap in frequencies $|\omega| \geqslant B$.
- Solutions with the same $B\omega$ are orthonormal on H with the measure $\sqrt{-h}d^3x = \sin(\theta)\, d\theta\, d\varphi\, dX$
- Global picture not yet clear. Spacetime - not globally hyperbolic. Given an arbitrary solution "in the small" - can it be extended to the whole spacetime?

Summary (d'Alembert equation):

Due to the remarkable fact, that the d'Alembert operator can be expressed as a linear combination of the Casimir operators of the symmetry group all solutions fulfilling the separation Ansatz can be determined explicitly. They are simple functions of $z = \tan(\frac{\theta}{2})e^{i\varphi}$.

arXiv:gr-qc/0703018 (soon to be upgraded)

Summary of the talk

- Rotating matter in GR leads to the appearance of “gravitomagnetic” fields.
- Geodesics in simple spacetimes with such fields correspond to trajectories of charged particles in spacetimes without these fields. Gödel solutions are spacetimes with a homogeneous “gravitomagnetic” component. These spacetimes are also homogeneous (though anisotropic), the geodesics “spirals” (in r, φ, X).
- A full family of solutions to the d’Alembert equation, expressible by elementary functions can be constructed.