Finite extension of stars (atmospheres) in the newtonian theory
Virial theorems

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1 General remarks

Clouds of fluids in the NT are solutions to the equations of Poisson

$$\nabla^2\varphi = 4\pi G\rho$$

and Euler

$$\rho\partial_i\varphi = -\partial_i p.$$

Here ρ is the mass density. In spherical symmetry we get from the Poisson equation

$$\varphi'(r) = \frac{4\pi G}{r^2}\int_0^r \rho(s)s^2\,ds.$$

Thus, the potential is monotonic with r. Let

$$\Gamma(p) = \int_0^p \frac{dp}{\rho(p)}.$$

If follows from the Euler equation that

$$\varphi + \Gamma(p) = \varphi_0,$$

where all functions are functions of r, and φ_0 is the constant of integration. All equations contain only derivatives of φ, and thus we may normalize the potential to vanish at infinity. The potential therefore statrts with a certain negative value at $r = 0$, grows monotonically with r, attains a certain value at the surface of the object (where $p = 0$; should such a surface exist), and finally grows as $-M/r$ towards $\varphi = 0$ at infinity. Because of that we may chose to regard p as an implicit function of φ, with φ_0 having the interpretation of the potential at the surface of the object, should it be present. If the object is infinite, than $\varphi_0 = 0$ and we have

$$\varphi = -\Gamma(p).$$

2 Virial theorem for interacting particles

Nonrelativistic interacting particles are subject to the following consideration: let

$$W = \sum_n \mathbf{x}_n \mathbf{p}_n$$

the long-time avarage of the time derivative of W, denoted by $< \dot{W} >= \frac{1}{T}\int_0^T \dot{W}$, vanishes for large T once we deal with a finite (in extension and in the number of particles) ensamble. On the RHS we get

$$RHS = \sum_n 2\langle p^2/2m\rangle + \sum_n \left\langle \mathbf{x}_n \frac{d\mathbf{p}_n}{dt} \right\rangle .$$

Suppose the particles interact with each other with the potential of the mutual interaction (of two of them) $V = 1/r^n$. Because of

$$x^i \partial_i^x V + y^i \partial_i^y V = \frac{-n}{r^{n+1}}(x^i \partial_i^x r + y^i \partial_i^y r) = (-n)V,$$

we get the virial theorem

$$0 = 2\langle E_K\rangle - n\langle E_V\rangle,$$

which for the Coulomb interaction gives $2\langle E_K\rangle + \langle E_V\rangle = 0$.

3 Virial theorem for a self-gravitating fluid

The following identity[1] is true

$$\partial_i \left[(x^j \partial_j \varphi + \varphi/2)\partial^i \varphi - (\partial_j \varphi\, \partial^j \varphi)x^i/2 + 4\pi G p\, x^i\right] = 2\pi G(\rho\varphi + 6p)$$

(the proof employs the Euler and Poisson equations; rescalling symmetry is the basis for the existence of this identity). Consequently, by integrating the above identity[2] we obtain the virial theorem for self-gravitating fluids

$$\int d^3x(\rho\varphi + 6p) = 0$$

for ideal, monoatomic gases this reduces to the identity derived previously for particles because $p(x) = \frac{2}{3}\, u(x)$, ($u(x)$ being the density of the kinetic energy of the gas), and the gravitational energy is just $\frac{1}{2}\int d^3x\, \varphi\rho$.

[1]Based on W. Simon, "On journays to the Moon by balloon", CQG '93.

[2]As to the surface terms: the object is by assumption of finite mass, which requires $\rho(R) \sim C/R^{3+\epsilon}$; the pressure must fall off faster because of the Euler equation; with $\varphi(R) \sim C/R$, it is necessary that $p(R) \sim C/R^{5+\epsilon}$ for $R \to \infty$. The other surface terms also vanish as a consequence of the required fall-off of the potential.

4 Finite extension of self-gravitating clouds of fluids

We shall give an indirect proof. Let us take the virial theorem for such fluids, and assume, that the cloud is infinite, $\varphi_0 = 0$. Then we should have

$$\int d^3x[-\Gamma(p)\rho + 6p] = 0$$

However, often the integrand is positive/negative at every point thus making the virial theorem impossible to be fulfilled, unless the object composed of such fluid is actually finite. The merit of this statement lies in the fact, that one can often judge the sign of the integrand right from the equation of state. For barotropic fluids

$$p = K\rho^\gamma$$

we get

$$-\Gamma(p)\rho + 6p = p\left(\frac{\gamma}{1-\gamma} + 6\right),$$

which is positive in many cases, e.g. for non-relativistic and ultra-relativistic degenerate fermi gases, as well as for adiabatic clouds composed of ideal gases. These fluids lead to clouds of finite extension. On the other hand for isothermal ideal gases we get no conclusion (such clouds are at best infinite (i.e. of finite, nonzero mass with infinite extension)).